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GUIDELINES FOR MATHEMATICS IN THE ELEMENTARY SCHOOL.
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DELAWARE STATE DEPT. OF PUB. INSTRUCTION, DOVER
PUB DATE SEP 66
EDRS PRICE MF-$1.00 HC-$8.84 219P.

DESCRIPTORS- *ARITHMETIC, *CURRICULUM, *CURRICULUM GUIDES, *ELEMENTARY SCHOOL MATHEMATICS, *MATHEMATICS, *TEACHING GUIDES, MEASUREMENT, TEACHING PROCEDURES, TEACHING TECHNIQUES, DELAWARE,

THESE GUIDELINES FOR CLASSROOM TEACHERS OFFER SUGGESTIONS FOR TEACHING ELEMENTARY SCHOOL MATHEMATICS IN A MANNER TO REFLECT RECENT CHANGES IN CONTENT, TECHNIQUES, AND APPROACHES TO TEACHING MATHEMATICS. THE PURPOSES OF THESE GUIDELINES ARE (1) TO DETERMINE A DIRECTION FOR MATHEMATICS EDUCATION IN THE ELEMENTARY SCHOOLS OF DELAWARE, (2) TO PROVIDE A COMMON BASIS FOR THE MATHEMATICS CURRICULUM FOR THE CHILDREN, (3) TO PROVIDE A SOURCE OF INFORMATION FOR THE PLANNING OF INDIVIDUAL DISTRICT PROGRAMS, (4) TO ESTABLISH CRITERIA FOR A BALANCED CURRICULUM THROUGH WHICH TEACHERS MAY EVALUATE THEIR OWN INDIVIDUAL PROGRAMS, (5) TO DEVELOP A LOGICAL SEQUENTIAL PROGRAM FOR USE IN THE ELEMENTARY GRADES, AND (6) TO ENCOURAGE USE OF MATHEMATICAL LANGUAGE. RECOMMENDED APPROACHES AND METHODS OF PROCEDURE ARE EXPECTED TO LEAD TO THE FOLLOWING GOALS--(1) PROVIDING FOR INDIVIDUAL DIFFERENCES, (2) STRESSING PRINCIPLES RATHER THAN SPECIFICS, (3) BUILDING THE STUDENT'S CONFIDENCE IN HIS OWN DISCOVERY ABILITY AND CREATIVE THINKING, (4) DEVELOPING THE STUDENT'S ABILITY TO ANALYZE VERBAL PROBLEMS AND TO TRANSLATE THESE INTO A FORM WHICH LEADS TO THEIR SOLUTION, AND (5) DEVELOPING AN ABILITY ON THE PART OF THE STUDENT TO COMMUNICATE HIS UNDERSTANDING. (RP)

Guidelines For Mathematics in The Elementary School

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September, 1966

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FOREWORD

Classroom teachers have indicated a need for guidelines to offer suggestions for teaching elementary mathematics, especially as the curriculum today reflects changes in content and approach.

Beginning in June, 1964, a committee was established to draft such guidelines for the elementary school. Separate committees of seven to fifteen members met for two weeks in the summers of 1964, 1965, 1966. To assure continuity, members on one year's committee were used on the next year's committee, where possible. This guideline was the combined result of the work of the three committees.

The guidelines for the intermediate level anticipate that students of average ability will begin the material by the beginning of grade four. Students of lesser ability would start at these levels later, whereas, those of greater ability may use the ideas as early as grade three. Arbitrary levels are indicated by letter. Levels sometimes overlap typical grade levels as indicated in the Grade Placement Chart.

Other topics that are appropriate for the elementary school, but which have not been developed in this guideline include:

(1) number bases
(2) interpreting and constructing graphs
(3) percentage problems
(4) ratio and proportion
(5) probability
(6) graphing of truth sets (one and two-dimensional)
(7) verbal problems involving inequalities
(8) functions

ELEMENTARY MATHEMATICS GUIDELINES COMMITTEE
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Mr. Morton Witlin, Absalom Jones School, Marshallton School District, Sixth Grade Teacher

PURPOSES OF THESE GUIDELINES

To set a direction for mathematics education in the elementary schools of Delaware.

To provide a common basis for the mathematics curriculum for the children of Delaware.

To provide a source of information for the planning of individual district programs.

To establish criteria for a balanced curriculum through which teachers may evaluate their own individual programs.

To develop a logical sequential program for use in the elementary grades.

To encourage accurate use of mathematical language.

To recommend approaches and methods that would lead to these goals:

1. Provide for individual differences.
2. Stress principles rather than specifics.
3. Allow for "open-endedness".
4. Build student's confidence in his own discovery ability and creative thinking.
5. Develop student's ability to analyze verbal problems and translate into a form which leads to their solution.
6. Develop an ability on the part of the student to communicate his understanding.

GOALS FOR THE STUDENT IN THE ELEMENTARY MATHEMATICS CURRICULUM

I. UNDERSTANDINGS

A. Patterns underlie mathematics.
B. A given problem can be solved by a variety of approaches and techniques.
C. The method we use is not the only method and perhaps not the best.
D. The reasonableness of methods of solution, and the solution itself, avoids illogical conclusions.
E. Mathematics is a method of communicating ideas and is a useful tool.
F. An understanding of mathematics is based on the sequential development of a few basic ideas.
G. Number systems, numeral systems and geometric figures are distinguished by their properties.
H. Efficient and accurate computation depends upon understanding, practice, and arrangement of symbols.

II. SKILLS

A. Facility in use of the four fundamental operations on whole numbers and rational numbers (fractional and decimal forms).
B. Ability to use common units of measure in measurement and computation.
C. Ability to perceive and identify geometric figures in many positions.
D. Ability to estimate within reasonable limits.
E. Ability to calculate mentally as well as in written form.

III. ATTITUDES AND APPRECIATIONS

A. Study of mathematics is fun.
B. Word problems are not difficult when you know how to translate them into mathematical sentences.
C. The various branches of mathematics (geometry, arithmetic, algebra, etc.) are inter-related.
D. Mathematics is a useful tool in all vocations and phases of life.
E. Mathematics is manmade and is always undergoing change.
F. Books and stories relating to mathematics offer stimulating reading.

6. List specific criticisms, commendations, suggestions for other approaches, and other comments.

General objective: Comment:

Guide Item No. Comment:

____

____

____

____

Use additional sheets for additional specific comments.

Please tear out this page and return to:

Richard R. Koch, State Supervisor of Mathematics
Department of Public Instruction, P.O. Box 697
Dover, Delaware 19901

TEACHER REACTION REPORT

TEACHER: Tear out this sheet, complete and return as indicated on page 6.

1. The detail of the guide is

excessive	______
good	______
insufficient	______

Comment:

2. Is the guide sufficiently specific as to content placement with respect to grade level?

yes	______
no	______

3. Is there any addition you would make to the guide? yes ___ no ___

4. Are sufficient specifics provided for remedial and enrichment cases? yes ___ no ___

5. What features of this bulletin do you find valuable?

GRADE PLACEMENT CHART

PRIMARY LEVEL

A - Grade 1

B - Grades 1 and 2

C - Grades 2 and 3

D - Grade 3

INTERMEDIATE LEVEL

E - Grades 4 and 5

F - Grades 4 and 5

G - Grades 4 and 5

H - Grades 4 and 5

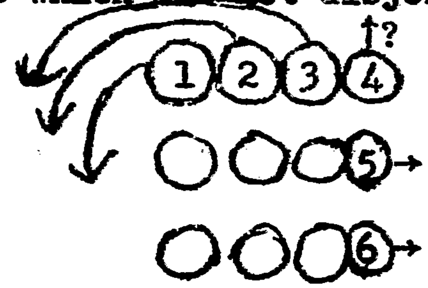
I - Grades 5 and 6

J - Grades 5 and 6

BASIC UNDERSTANDINGS AND SUGGESTED TECHNIQUES

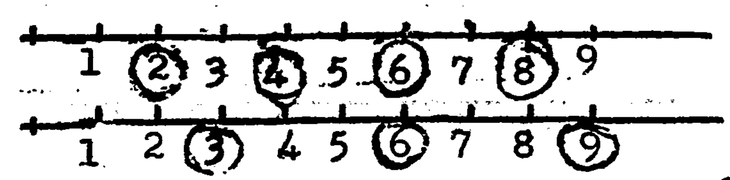
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
NUMBER What is number? A-1 What is set?	A sst is a collection (group) of objects.	Build a basic vocabulary.	Point out or discuss various sets, including sets of many elements (such as set of students; set of dishes; set of desks; set of books) and sets of a few elements (such as the set of shoes John is wearing; a set of doors in the room; set of teachers desks in the room; set of Mary's noses). Include examples of sets where the objects are different, such as a set containing a pencil, a piece of chalk, and a piece of paper.
A-2 Equivalent sets.	Sets are equivalent if we can set up a one or to one correspondence. (matching)	Build readiness for concept of number.	Match 3 or 4 pupils with an equal number of pencils in the teacher's hand, with chalk board erasers, chairs, or books. Match a larger number of pupils with sets containing an equal number of objects. To illustrate one to one correspondence, each pupil is matched with a single object from the second set. Avoid the use of specific numbers in a reference to the sets or in the matching process. (ILLUSTRATION) The teacher might arrange sets of 3 balls, 4 books, 5 erasers, 4 pencils, 4 crayon boxes, 5 rulers, 1 scissors and a mixed set of 3 objects and ask a student to pick out two equivalent sets.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
A-3 Equal Sets	Sets are equal if they contain exactly the same objects.	Distingush between equal and equivalent sets. (Vocabulary development)	Illustrate by examples such as a set of first graders in this room is equal to the set of boys and girls in this room under 10 years old. (We are looking for two different ways of naming exact-ly the same set. Two <u>different</u> sets of chalk each with 12 pieces are equivalent sets but not eq equal).
A-4 Ordering Sets	A set is greater than a second set if it has objects left over after matching; a set is less than a second set if it does not have enough objects to match the second set.	Establish concept of more than and less than.	Take the sets cuch as in A-2, compare two at a time and finally arrange in order from smallest to the largest.
A-5 Assigning number names (1-9)	Sets that are equiva-lent to each other have the same cardi-nal number. Set is the basis of number.	Number names we use have been arbitrarily assigned for ease in communication.	Names are assigned by custom. Use sets of concrete objects and charts picturing sets which the child can recognize at a glance (without counting) and ask students to reply with the assigned number. (Recall stage).
A-6	The order of the num-ber names follows the order of the sets(that were previously deter-mined).(Step A-5).The order so determined is one,two,three,four,etc.	Introduce the proper sequence of number names to nine.	Use sets of concrete objects and charts to show sets in order of size and apply the number names to these sets. (<u>One</u> through <u>Nine</u>).

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
A-7 Assigning numerals to number names, recognition and association of the numeral with corresponding sets.	The numeral is the written symbol for the number idea. The student must recognize numerals 1 thru 9, relate them to the corresponding number.	Establish the recognition of written communication. **Introduce number line to reinforce order.**	Take examples of concrete sets or charts illustrating such sets. The teacher writes or points to the numeral assigned to the respective sets. **Draw a number line. Mark points and label. Point to other places and ask what number belongs there.** 2 4 
A-8 Learning to write numerals.	Develop the visual pattern needed to construct the numeral.	Write the numerals.	Illustrate by example the writing of each numeral. Discuss starting point and order of constructing each numeral; students trace printed numerals, copy without tracing, and finally write without model before them.
A-9 Disjoint Sets	Disjoint sets have no objects in common. (The same object is not in both sets).	Introduce disjoint sets which will be used in addition. (The case of disjoint sets is the only case where addition will apply)	Ask students in the first row to stand on one side of the room and all students in the last row to stand on the opposite side of the room. Note that no student is in both sets. This is an example of disjoint sets. Everybody sits down. Have the students in the first column stand on one side of the room and those in the first row on the other. Somebody does not know which side to go because he belongs to both sets. This is an example of sets which are not disjoint. 

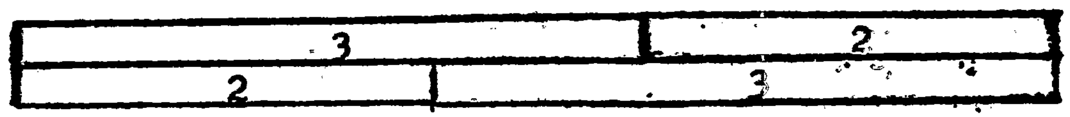
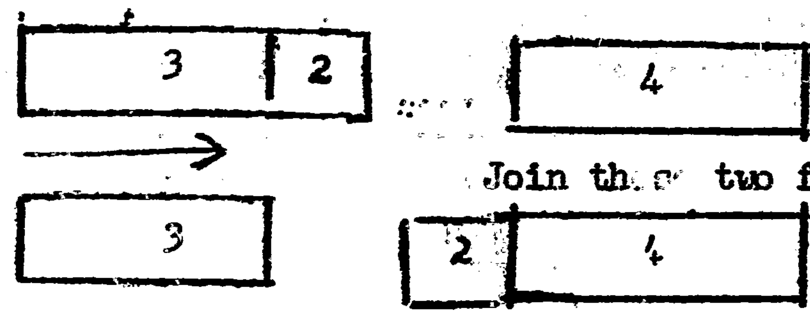
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
OPERATION A-10 Union	Union applies only to sets. Operation of union involves putting sets together to form one set. The idea of union without regard to specific numbers is to be developed.	Illustrate concrete examples from which addition will be derived.	Have the students seated around the table (or in one row) stand. Have the students around a second table (or in another row) stand and join the first group forming one larger group. Be sure to choose disjoint sets (sets which have no youngster in common). Do not refer to the particular number of objects in the respective sets.
A-11 Addition	The cardinal number of the union set is the sum of the cardinal numbers of the sets joined. The operation on numbers to form the sum is addition. Addition follows from the operation on sets called union, but addition and union are not synonomous.	Define addition.	Repeat the concrete examples as in A-10 and discuss orally using numbers. The number of the first set is three; the number of the second set is two. Pointing to the combined set, say, "The sum is five." (xxx) 3 (xx) 2 SUM (xxxxx) 5
A-12 Addition Notation	Addition can be expressed symbolically. (+, =)	Establish written communication.	Use example as in A-11 and discuss as, "three plus two equal five", and write as 3 + 2 = 5, or $\begin{array}{r}3\\+2\\\hline 5\end{array}$
A-13 Addition Combinations	Numbers have many names.	Develop student confidence in discovery for himself. Allow for	Give each student five sticks and ask them to illustrate the mathematical sentence 3 + 2 = 5. Then ask the student to regroup the five sticks to form other combinations

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
		flexibility and multiple approaches for later developments. Also use number line introduced in A-7 to illustrate addition.	without the teacher specifying which ones. Permit and encourage combinations such as 2 + 2 + 1. For each combination have the student write the sentence showing the new name for five: $5 = 3 + 2$ $5 = 1 + 4$ $5 = 4 + 1$ $5 = 2 + 1 + 1 + 1$ $2 + 3 = 5$
A-14 Number line	A number line can be used to indicate the order relationship of numbers.	Introduce a tool for later use.	Have a student write the numerals from 1 to 9 in order, draw a line beneath, mark dots on the line under the respective numerals, and associate the numbers with the respective dots.
A-15 Counting by multiples	Counting patterns need not utilize all numbers.	Build confidence in and extend counting. Prepa. for multiplication.	Have a student repeat the number of the set at each step as sets of two pencils are laid on the table. If necessary, have a student point to every other number on the line and read the number. Have a student recite the numbers by twos without using sets of objects. Repeat with threes.



by 2's

by 3's

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
A-16 Counting backwards	Counting patterns can be followed in either order.	Prepare for subtraction.	Start with a set of 9 books. Ask for the number of set. Ask for the number of the set as books are taken away, one at a time. Start with 9 on the number line, read the number immediately before 9, read the number immediately before 8, etc. Have students recite numbers backwards without visual or concrete aids.
A-17 Commutative property of addition.	One pattern in the addition combinations is commutativity.	Introduce for later use in building combinations.	If $3 + 2 = 5$, and $2 + 3 = 5$, then $3 + 2 = 2 + 3$. Does this exist in other combinations? This should be a natural out-growth of A-13. Also use a concrete approach such as Cuisenaire rods: 
A-18 Associative property of addition.	Another pattern in addition is associativity.	For later use in regrouping. (Carrying)	$(3 + 2) + 4 = 3 + (2 + 4)$ because each is another name for 9. Parentheses mean "do me first." The student should be given the opportunity to discover the pattern first. Use rods or sets of objects to clarify Join these two first  Join these two first

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLE
A-19 Regrouping	The associative property is useful in renaming groups.	Practice in using the associative property.	Illustrate by steps: 1. $5 + 3$ — Student does not know this combination. 2. $5 + (2+1)$ — Since $2 + 1$ is another name for 3, we may replace. 3. $(5 + 2) + 1$ — The associative property allows regrouping. 4. $7 + 1$ — The student knows $5 + 2$ previously. 5. 8 — The student knows $7 + 1$ previously. 6. $5 + 3 = 8$ — Therefore 8 is a simple name for $5 + 3$.
PROBLEM SOLVING A-20 Oral Number Stories (Addition)	A mathematical sentence tells many stories.	Show that mathematics is a short way of communicating ideas.	Write $2 + 3 = 5$ on the board and ask children to make up a story using this fact. Teacher or children may give the story and ask the class or a child to write the number sentence.
NUMBER A-21	When a set contains no objects it is said to be "empty".	Give a visual meaning to the number zero. To provide readiness for subtraction. To introduce zero.	Illustrate by showing a jar containing milk money. The money is sent to the cafeteria. How many coins are in the set in the jar now? None. Zero is the cardinal number for the set of coins. Ask where zero would be placed on the number line.

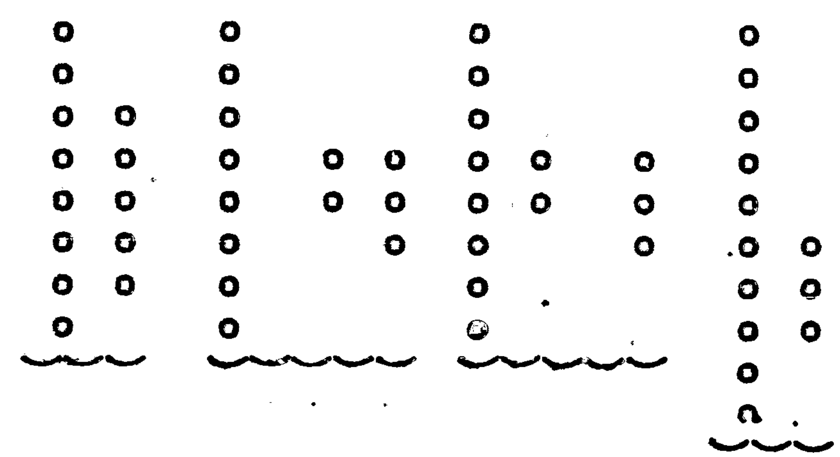
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
OPERATION A-22 Subtraction Formation of subsets.	One method to form a subset is to remove a number of objects from one set to form another set. The set of objects left is called the remainder set.	Build a concept on which to base subtraction.	Have one row of children stand up making a set. Tell some children of the set (making a subset) to go out of the room. The children left standing are the remainder set. To reenforce empty set - all children could sit down. How many are in the set of children standing? None. The remainder set is the empty set.
A-23 Subtraction	The operation on numbers to find the difference is subtraction.	Definition.	Repeat the concrete example of A-22. Have the children give the cardinal numbers of the first two sets and have them determine the number of the remainder set. The cardinal number of the remainder is the difference.
A-24 Introduce subtraction notation.	Subtraction can be expressed symbolically (-, =)	Establish written communication.	Use the example in A-22. Discuss as "six subtract (or minus) two equals four" and written as $6 - 2 = 4$ or $\begin{array}{r} 6 \\ \underline{-2} \\ 4 \end{array}$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
A-25 Subtraction combinations.	Knowing the combinations permits ease in computation.	Discovery. Flexibility	Use concrete objects as in A-13. (Addition) Other number names for 5 include: 9 - 4 6 - 1 8 - 3 5 - 0 7 - 2
A-26 PROBLEM SOLVING Oral number stories.	A mathematical sentence tells many stories.	Show that mathematics is a short way of communicating ideas. One of the aims in problem solving should be the development of reasonablieness. For example, the child should understand the remainder set cannot be greater than the first set.	Use method as in A-20. If a set of 6 bluebirds is washing in a birdbath and two fly away, would 7 be a <u>reasonable</u> answer for the number remaining when there were only 6 to start?
A-27 <u>GEOMETRY</u> Shape A-three dimensional objects. B-Two dimensional objects.	Objects have different shapes. These shapes have names.	Recognize and name these shapes. A-ball, cube, rectangular solid. B- circle, square, rectangle, triangle.	A-Handling and discussing. B-Use only paper figures and drawings on paper and flannel board cutouts. Avoid three dimensional objects such as books which will implant or convey ideas other than the two-dimensional ones desired.

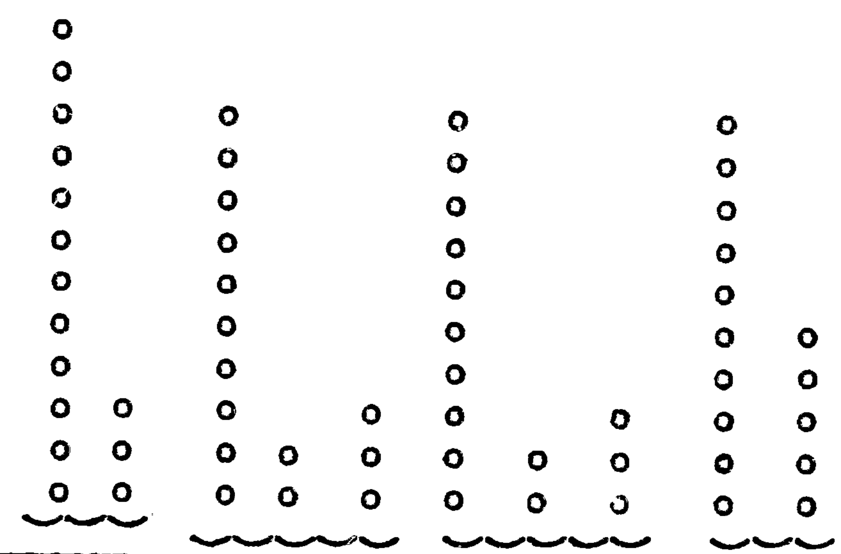
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
A-28 Size	Geometric shapes have various sizes.	Develop the concept of comparison and extend vocabulary.	Use concrete objects and drawings with comparative words. (small, smaller, long, short, etc.)
MEASUREMENT A-29 Comparative terms.	Many words are necessary for describing relationships.	Build readiness for measurement, develop concepts and vocabulary.	Take concrete objects and describe them in terms of heavy or light, big and little, etc. Also work with such terms as before, after, between, above, below, yesterday, today, tomorrow, warmer, colder, left, right, etc.
A-30 Time	Time has many divisions.	To make the child aware of the larger divisions.	Teacher should use names of days, months and such terms as morning, etc. in meaningful situations.
A-31 Money	Pieces of money have value.	To recognize penny, nickel and dime.	Use real money for identification purposes.
A-32 Temperature	The thermometer is used to measure temperature.	Readiness for reading thermometers.	Reference to thermometer. Reference to higher and lower, hotter, colder, etc.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
Level 2 B-1 NUMBER Numbers greater than nine.	There are numbers greater than nine.	Extend the number system. (Build appreciation.)	Ask the students if there are sets whose cardinal number is greater than nine. Have the student name some examples.
B-2 Assigning number names ten thru nineteen.	New names are needed for the new numbers.	Extending vocabulary for communication.	Take a set of nine children, add an additional child, indicate a need for a name for the cardinal number, give it a name (ten) add an additional student and repeat, continuing thru nineteen.
B-3 Place Value	To avoid ten new numerals a grouping idea is needed.	(a)-Motivate need of place value.	(a)- Ask the student whether new numerals would be needed for each of the new number names. This many new symbols would be difficult to remember. Is there an approach where we could use the old numerals as symbols to form new numerals (such as ten) rather than inventing a new design such as Λ? Suggest forming a group. Ask for suggestions for the size of the group to use.
		(b)-Apply grouping to the sets to develop new numerals.	(Oral) Consider the set of ten objects as a "group", the set of eleven objects as a "group and one", a set of twelve objects as a "group and two"... Write in a list as: 1 group 1 group + 1 read as one group and one, etc. 1 group + 2

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
			Have the students look for a pattern in this list and ask them for suggestions for completing the pattern: 1 group + 0 Using the same technique expand to "2 groups + 0", "3 groups + 2", ...Teacher may elect to take each number in order; or work with decades first, then fill in between the decades; or take numbers at random including decades. (c) - Repeat the process but use "1 ten" instead of "1 group".
B-4 Place Value Notation.	(a) There is a short way of writing the place value idea.	(a) Introduce common notation for communication.	(a)- Write in a list next to the list developed in B-3, (c) the numerals 10,11,12...to show this shorter form of notation. 1 group 10 1 group + 1 11 1 group + 2 12 etc.
	(b) The column in which a digit is written determines whether it represents ones or tens.	(b) Reinforce Place value.	(b) Ask the students what would happen if we forgot to write the zero in a numeral such as 10,20,etc. The teacher familiar with the history of number might bring in the Babylonian System which had no zero.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
B-5 Counting backwards and by twos.	Previous patterns for counting extend into the new numbers.	Readiness for multiplication and division.	Have student count forward as far as he can, putting a pencil on the desk as he counts one and adding a pencil each time he counts. Now take a pencil away each time as he counts backwards. Start with numbers other than one and count forwards or backwards. Have students count hands (by twos) as students rise one at a time; also backwards as students sit down.
B-6 Combinations	Numbers have many names.	Regroup in useful directions. Convert sums into other sums in which one addend is ten to prepare for column addition.	Illustrate with bead frame: 8 + 5 =  $8 + 5 = 8 + (2 + 3)$ $= (8 + 2) + 3$ $= 10 + 3$ $= 13$

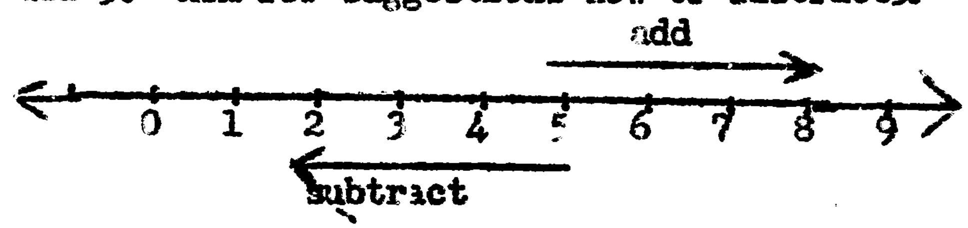
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVE	TEACHING APPROACH AND EXAMPLES
B-7 PROBLEM SOLVING Verbal Problems	A mathematical sentence tells many stories.	Show that mathematics is a short way of communicating ideas.	Have the children give stories for $8 + 3 = 11$, $5 + 4$ is less than 10. Introduce the symbol $<$ (less than) and rewrite as $5 + 4 < 10$.
B-8 Missing Addend	Subtraction may be considered as looking for the missing addend.	Multiple approaches to subtraction. Build confidence in discovery.	Ask student to find the number or number which make mathematical sentences true. The teacher should not supply the answer. If there is no student response, let it drop for a few days or weeks and return when the children are ready. $4 + \square = 7$ $\triangle + 6 = 9$ $2 + 3 = \square$ $\triangle + \square = 7$ The teacher should avoid using a sum that would make both the $\triangle$ and $\square$ the same number such as in $\triangle + \square = 6$.
B-9 Properties of Zero	A number plus zero equals the number. Some sentences have more than one number that makes the sentence true.	Test knowledge of zero to this point.	$4 + \square = 4$; $\triangle + 0$ (zero) $= \triangle$ Find numbers that would make these sentences true. Then make stories from the sentence.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACH AND EXAMPLES
B-10 Written story problems.	Stories can produce mathematical sentences. A mathematical sentence is a shorter way of writing a number story.	Show the relationship of mathematics to story problems.	A child gives a story and the teacher writes it on the blackboard. Children read the story and write the sentence for it. Children can write their own stories and mathematical sentences.
B-11 OPERATION Addition One two-digit addend. (No carrying)	Multi-digit numbers can be added by regrouping.	Extend skills of addition computation.	STEP I $12 + 3 = (10 + 2) + 3$ $= 10 + (2 + 3)$ $= 10 + 5$ $= 15$ STEP II $\begin{array}{r} 10 + 2 \\ + 3 \\ \hline 10 + 5 \end{array}$ STEP III $\begin{array}{r} 12 \\ +3 \\ \hline 15 \end{array}$ Use the bead frame to show 12 + 3 = 
B-12 Two two-digit addends (no carrying).	Multi-digit numbers can be added by regrouping.	Extend the skills of addition computation.	$\begin{array}{r} 13 \\ +25 \\ \hline \end{array} \begin{array}{c} = \\ = \\ \end{array} \begin{array}{r} 10 + 3 \\ 20 + 5 \\ \hline 30 + 8 = 38 \end{array}$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACH AND EXAMPLES
B-13 One two-digit addend (with regrouping to teach carrying).	Some addition requires regouping to form decades in order to write the sums.	Enhance ability to increase computational skill.	Use the bead frame to show as in B-11: $\begin{array}{r} 14 \\ +\ 7 \\ \hline \end{array}$ $\begin{aligned} 14 + 7 &= (10+4)+7 \\ &= 10+(4+7) \\ &= 10+11 \\ &= 10+(10+1) \\ &= (10+10)+1 \\ &= 20+1 \\ &= 21 \end{aligned}$ This method of presentation helps show child that 11 should not be written in the ones column. Both the horizontal and the vertical should be developed side by side. In the 5th row, (10+10)+1 is read as (1 ten + 1 ten) + 1. The sum is 2 tens not 20.
B-14 Addition Matrix	A matrix arrangement will help show patterns.	To use matrix to show patterns in addition combinations whose sums are not yet known. To introduce reading of tables.	Demonstrate matrix arrangement to summarize known addition combinations. $\begin{array}{c\|cccccccccc} + & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$ Ask students to fill in sums of combinations that are known. Raise questions to indicate many are unknown.

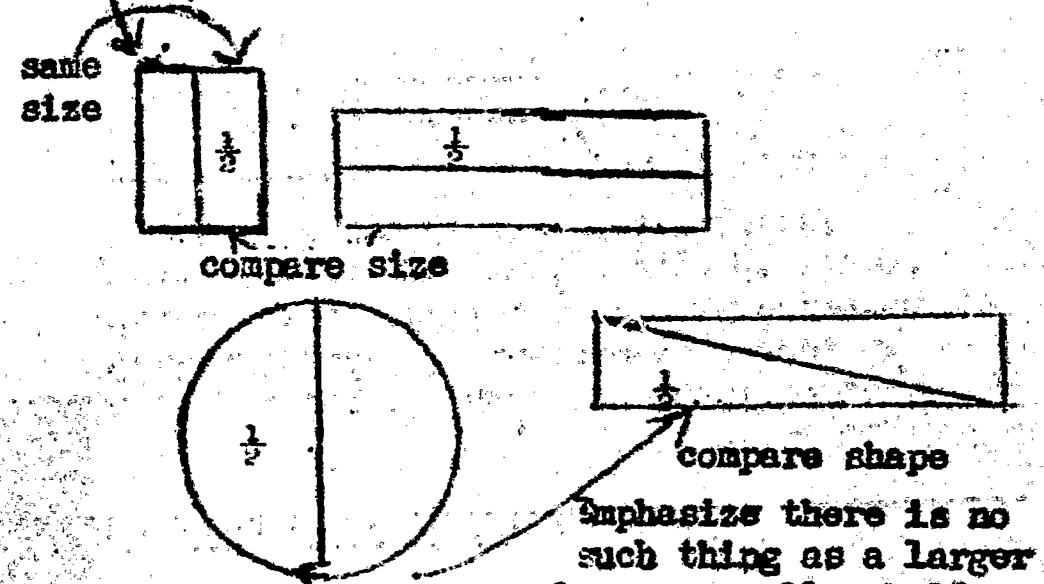
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
B-15 Two two-digit addends with carrying in one place.	Some addition requires regrouping to form the decades in order to write the sum.	Enhance ability to increase computational skill.	STEP I: $23 + 49 = (20 + 3) + (40 + 9)$ $= (20 + 40) + (3 + 9)$ $= (20 + 40) + 12$ (do 3 + 9 first) $= 60 + 12$ $= 60 + (10 + 2)$ $= (60 + 10) + 2$ $= 70 + 2$ $= 72$ Use the bead frame to illustrate as in B-11. STEP II: $23 + 49 = \begin{array}{r} 20 + 3 \\ \underline{40 + 9} \\ 60 + 12 \end{array} = 70 + 2$ Alternate approach: $\begin{array}{r} 23 \\ \underline{+49} \\ 12 \\ \underline{60} \\ 72 \end{array}$ $\begin{array}{l} \\ \\ = 3 + 9 \\ = 20 + 40 \\ \\ \end{array}$ (read as two tens plus four tens)

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
B-16 Disjoint Sets	Addition sums are meaningless if the sets joined are not disjoint.	To recognize limited use of addition.	Have students in first row stand. Find the cardinal number of the set. Have students in the first column stand. Find the cardinal number of this set. Find the sum of the two cardinal numbers. Compare this number with the cardinal number of the union set obtained when the students in both the first row and first column stand.
B-17 Number line addition.	The number line can be used to illustrate addition.	To allow for multiple approaches.	To show 9 + 7, ask a student to draw an arrow on the number line illustrating 9. (The arrow should start at 0, end at 9). Repeat for 7. Ask for suggestions to show 9 + 7. (Arrow for 7 should start at 9. Where arrow ends indicates the sum.
B-18 Commutative Property.	The commutative property can be used to check addition or to select an easier method.	Illustrate possible checking procedure. Show application of commutative property.	Ask for student reactions and observations to the following chart or boardwork: 9 + 8 = 17; 8 + 9 = 17; 24 + 31 = 55; 31 + 24 = 55; 28 + 15 = 43; 15 + 28 = 43 78 + 18 = 96; 18 + 78 = 96; 56 + 3 = 59; 3 + 56 = 59; 58 + 19 = 77; 19 + 58 = 76 Expected reactions: The sum is the same regardless of order. Unlike sums in the last pair indicates an error without actually adding to prove it. If sum is uncertain reverse the order of the addends and see if the sum is the same.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
B-19 Subtraction. Inverse idea.	Subtraction is the inverse of addition. Addition is the inverse of subtraction.	Relate the operations of addition and subtraction.	(A set of) 5 birds are eating corn and a (set of) 3 birds fly down to eat also. $5 + 3 = \Delta$. Three birds fly away. $\Delta - 3 = 5$. Summarize as $(5+3) - 3 = 5$. same Adding a number followed by subtracting the same number brings back the original number.
B-20 Subtraction Separation	In some subtraction problems the subset is separated but not physically removed.	Show the separation idea of subtraction.	1. Take a set of children. Ask the boys to form a subset. To do so the boys had to be separted from the original set of children. (Do not assign or discuss the cardinal number of the sets). 2. Take a set of 10 chairs. Separate 6 chairs. How many chairs are in the other set? Ask children to write the mathematical sentence for this operation: $10 - 6 = 4$ or $6 \quad 4 = 10$ (Read as: Ten subtract six equals four.) not read "–" (the subtraction sign) as "take away."
B-21 Subtraction with the number line.	The numberline can be used to illustrate subtraction.	Present multiple approaches.	Draw a line, indicate 5, ask children how to add 3. Ask for suggestions how to subtract3. 

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
3-22 Subtraction Combination	Review	Re-enforce skills.	Have students write other names for numbers through nine such as 7. Include names such as $7=9-2$.
3-23 Subtraction Combinations.	Ten through nineteen have related subtraction combinations. (other number names)	Build skills for subtraction. Allow for multiple approaches.	Use several approaches such as frame arithmetic. ($13 = \triangle + \square$; $11 = \triangle - \square$) : number line: other names for a number ($12 = 15 - 3$; $12 = 17 - 5$). Include vertical notations $\begin{array}{r}18\\ -9\\ \hline\end{array}$
3-24 Subtraction with zero	When zero is subtracted from any number, the difference equals the number. When a number is subtracted from the same number the difference equals zero.	Build the properties of zero.	Find numbers which make the following true: $\triangle - 0 = \triangle$; $\square - \square = 0$. Avoid the use of the circle as a place holder for a numeral because of the confusion with zero. Have students generalize from the mathematical sentences to evaluate their understandings.
3-25 Addition Matrix.	The addition matrix can be used in subtraction to find the missing addend.	To build skill in reading tables. To allow for multiple approaches.	Have students complete the addition matrix. To find the numbers which make the sentence $8 + \triangle = 15$, students are asked to devise a method using the matrix. The known addend, 8 gives many possible sums in the column beneath. The known sum, 15 limits the other possible addend to one, indicated in the appropriate row.

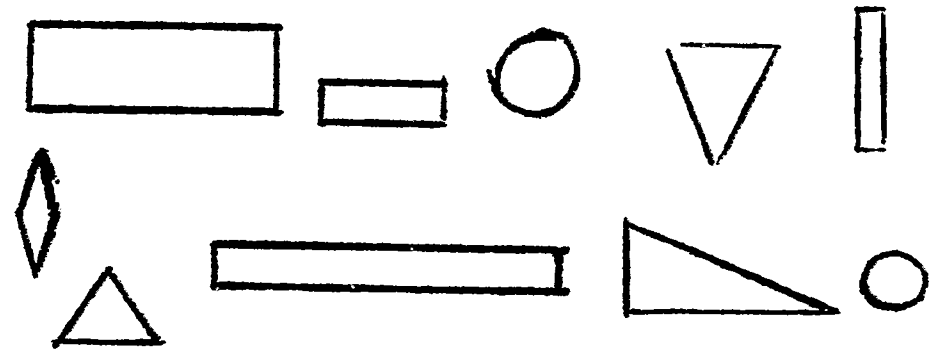
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
PROBLEM SOLVING B-26 Solving Oral and Written Problems.	Stories can be expressed as mathematical sentences. When stories include two unknown numbers which may be different (they may turn out to be alike), different symbols must be used to write the sentence. When the problem states that the unknown must be the same number, the symbols must be the same.	To apply story problems to the more difficult combinations and to two variables (like and unlike cases.)	Example with two variables: John had 18 marbles. Some were red and some were blue. Give many combinations. $\Delta + \square = 18$ (Δ and $\square$ can be like numbers.) Example with like variables: Jack and Bill both have the same number of marbles. They were all put in the same bag. There were 16 marbles in the bag. How many did each boy have? $\Delta + \Delta = 16$ Have students make up stories to fit given mathematical sentences. Children are now ready to write their own number stories and read stories that are written.
NUMBERS B-27 Common Fractions	There are numbers other than whole numbers. One kind can be written as fractions. When dividing a whole object the parts must be the same size. (The shape may depend upon choice). A fractional part such as $\frac{1}{2}$ may vary in size and shape depending upon the original whole.	Extend the number system.	Manipulate materials to show that they have parts. (Folding paper to form halves and quarters, etc.)

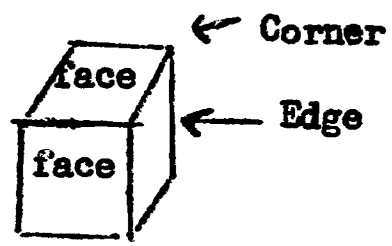


Emphasize there is no such thing as a larger half of an apple or a smaller half.

Both parts of the same object must be the same size to call it half.

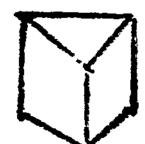
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
GEOMETRY B-28 Classification of simple figures (a) 3 dimensional (b) 2 dimensional	Objects and figures can be classified in different ways.	Recognize and compare geometric figures.	Display a set of large and small balls, blocks, boxes, closed oatmeal boxes, closed jars, pencils, etc. Ask children to place them in groups. (By size or shape) Repeat with two dimensional figures:
B-29 Geometry	Parts of solids have names such as face, corner, and edge.	Give names to the dimensional figures.	Use objects having different shapes. Discuss the name and number of faces, corners, edges on a rectangular solid; cube; pyramid; triangular prism. These should be hollow figures to show inside, outside.

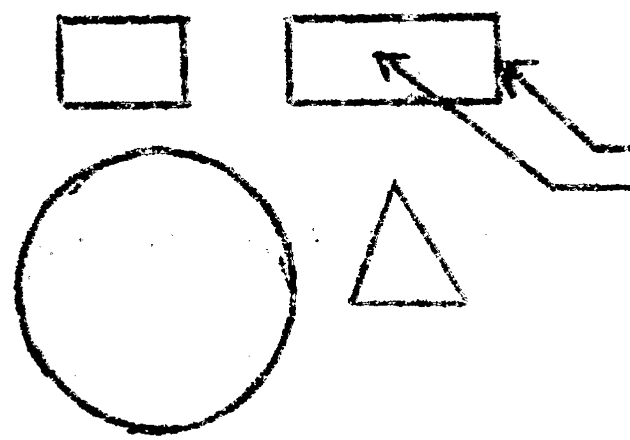




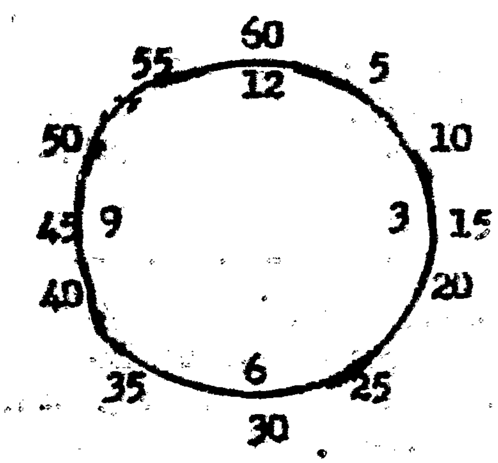
Box (a rectangular solid)


pyramid

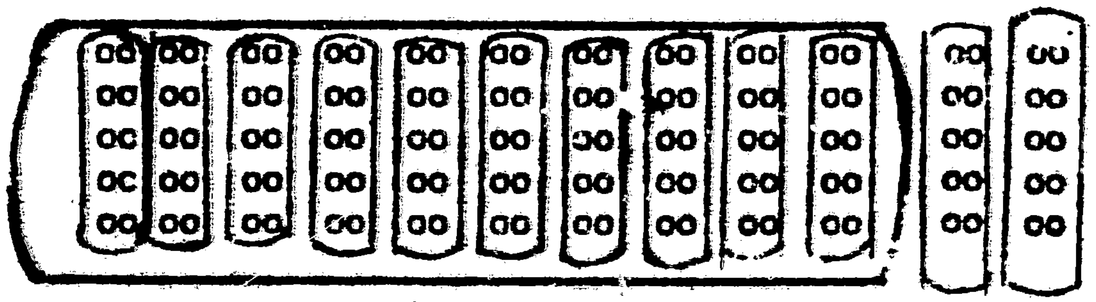

triangular prism

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHER APPROACH AND EXAMPLES
8-30 Two-dimensional figures.	The figure is only the border. This is not to be confused with its interior.	Distinguish between perimeter and area.	 Can be taught by having student color border blue, interior yellow, etc. The two together form a region called square region, triangular region, circular region, etc.
8-31 MEASUREMENT. Comparative terms for 3-dimensional objects.	Objects can be arranged in order of size.	Order objects by size.	Take some cubes or balls of varying sizes and ask a student to arrange from smallest to largest. Discuss as: a baseball is smaller than a basketball. Repeat in opposite order (larger to smaller); use the language "larger than".
8-32 Congruence.	Figures that are alike in shape and size are congruent.	Introduce congruence for use in measuring.	Ask students to order some sets of circular regions when the set happens to contain some congruent ones. This should raise the need for a word such as congruent, to describe this relationship. DO NOT use equal in this relation. Repeat with balls and blocks. Reserve "equal" for names for the same thing.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE OR OBJECTIVE	TEACHER APPROACH AND EXAMPLES
B-33 Simple measuring.	Measure is the number of units needed to fill a region. (The interior of a three dimensional figure with the figure itself is a region, called a space region.)	Build the concept of measurement.	Supply a set of different sized blocks. Let the children experiment to determine the number of blocks needed to fill a given box. Note that the student picked congruent blocks or discuss if he did not. Indicate that the number of congruent blocks is called the measure. Repeat by counting the number of congruent square regions needed to cover a larger square region. Pick a square that comes out even.
B-34 Value	A penny has a value of 1 cent; a nickel - 5 cents; a dime - 10 cents; a half-dollar - 50 cents. Quarters (25¢ piece) and half dollars (50¢ pieces,) have equivalent combinations of the other coins.	Become familiar with the value of the common coins.	Exchange teacher's coins for equivalent student coins. (lunch money) Challenge the students to determine the greatest equivalent combinations for 25¢. Find two ways of making 30¢ with 6 coins.
B-35 Dollars and parts of dollars.	A dollar has a value of 100 cents. Two 50¢ pieces have a value equal to one dollar and are therefore half-dollars. Repeat for 25¢ pieces.	Relate fractions to money.	Use real money and discuss the relationships.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHER APPROACH AND EXAMPLES
8-36 Time Hours	Time may be divided into hours.	Teach child to tell time to the nearest hour.	Have a number line that goes to 12 with same length as the clock circumference. Have zero at the top and wrap it around the clock. Take the clock face, move hands to different hours. Read and write as 2 o'clock.
8-37 Minutes	Hours can be divided into minutes.	Teach child to tell time with minutes.	Have a number line marked in fives to sixty and wrap it around the clock. Teach concepts of minutes after such as: 2 hours and 20 minutes, 3 hrs. and 50 minutes, etc. Go to conventional language as two twenty, three fifty. Write as 2:20, 3:50. Repeat with regular clock face. 
8-38 Temperature	Reading temperature is accomplished by reading a number line.	Read temperature.	Associate a number line to an expanding column of mercury. Read the numbers and record adding a degree sign. If the idea of a below zero does not come in as a childhood experience it can be presented by the teacher.

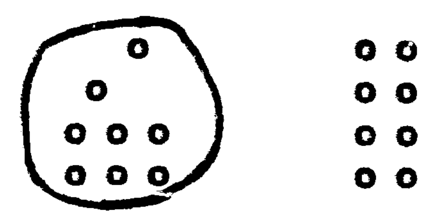
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHER APPROACH AND EXAMPLES
B-39 Statistics	A bar graph facilitates a study of pattern such as changing temperature. Graphs can be used for predicting unknown values.	Introduce prediction and graphing.	Hand out duplicated sheets which show thermometer diagrams. Color diagrams in order for readings either each hour or same hour each day. Note patterns of change. Have students predict or guess the reading for the next hour or day based on the pattern in the graph. Check with an actual reading when the time comes.
B-40 NUMBER Place Value	Sets larger than ten 10's necessitate larger groupings.	Extend place value to 3 positions.	Review that the limit of ten symbols necessitated a grouping idea to express as a numeral number larger than ten. Question: can we now write any size number? How could twelve tens be expressed with the symbols available? Show the need for larger size groups when writing numerals. Have students determine at which point the larger groups are needed. Use diagrams to emphasize collecting groups of ten to form groups of a hundred.

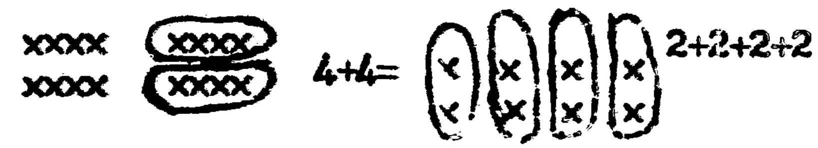
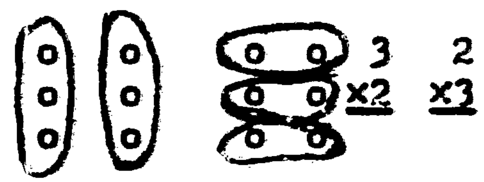
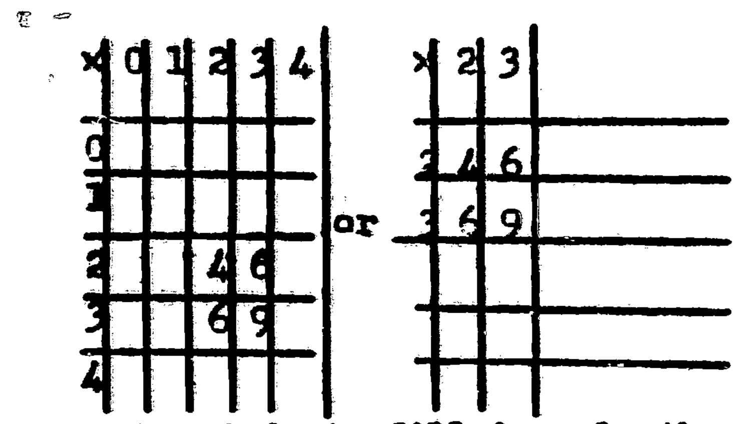


One large group plus two small groups express in short notation as:

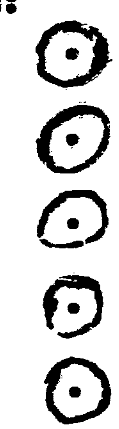
1 2 0

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHER APPROACH AND EXAMPLES
B-41 Place Value	Sets larger than ten 100's necessitate larger grouping.	Extend place value to 4 positions.	Have students determine if larger groups will be needed to express all numbers and if so, have students predict or determine the size of the next larger group needed based on the inadequacy of available symbols to express the number.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENT	PURPOSE AND OBJECTIVES	TEACHING APPROACH AND EXAMPLES
C-1 OPERATIONS Multiplication as shortened addition.	Multiplication is a short way of expressing repeated addition of the same number.	Introduce multiplication notation.	Illustrate with an addition problem and show a shorter way of writing it: (a) $\begin{array}{r}2\\2\\2\\+2\\\hline\end{array}$ (b) $\begin{array}{r}2\\\times4\\\hline\end{array}$ (c) 2+2+2+2= (d) 4x2= Read vertical notation (b) as "add 2 four times", the horizontal notation (d) as 4 two's. (Avoid using products above 9.)
C-2 Multiplication in terms of sets.	Multiplication is the operation of obtaining the cardinal number of the union set by joining equivalent disjoint sets. The cardinal number of the union set is called the product.	Interpret multiplication in terms of sets.	Use four different sets of two objects each. Put these sets together (the children will quickly realize that ordered set arrays are easier to work with.  After they are put together, ask the student for the cardinal number of the resulting union set and call this number the product.

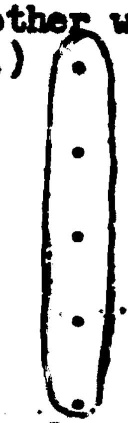
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
C-3 Relationships in multiplication.	The ordered array, the set approach, and repeated addition approach are inter-related.	Relate three interpretations of multiplication.	This product could be interpreted as a 2×4 array or as 2 sets of four, or as the sum of 2 fours; or, as 4 sets of two or the sum of four two's. 
C-4 Commutativity	The cardinal number of the resulting set is the same regardless of whether the array is considered horizontally or vertically.	Introduce the commutative property of multiplication.	 This can be written as 2×3 or 3×2
C-5 Matrix	Patterns in the combinations can be studied through the rule of the matrix.	Summarize the combinations.	 Have the students fill in only the combinations which they have previously developed. If the student extends the pattern, then permit him to expand the matrix.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES
C-6 Factoring	A factor is either of the numbers we multiply to form the product.	Practice combinations and learn new vocabulary.	$\Delta \times 2 = 8$ $\quad 2 \times 4 = \Delta$ $3 \times \square = 9$ $\quad \square \times \Delta = 6$ Discuss the first problem; look for the missing factor, repeat with others.
C-7 PROBLEM SOLVING Verbal problems.	Certain situations can be expressed in terms of multiplication.	Relate verbal problems in terms of mathematics.	If you have 2 rows of 4 chairs, how many chairs are there? Take an example such as $\Delta \times 2 = 8$ and have the student make up stories.
C-8 Associativity	Associativity holds for the operation of multiplication.	Develop the number properties.	Consider a set of buttons each having two holes.  Ask the student how many holes are in the complete set and ask the pupil for his procedure. Some will determine the number of holes in a row (3x2) first while others will find the number of buttons first (4x3). The result is the same, stated: $4 \times (3 \times 2) = 24$ $(4 \times 3) \times 2 = 24$ Therefore $4 \times (3 \times 2) = (4 \times 3) \times 2$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
			Alternate approach. Illustrate the set of buttons above and interpret (write) as four rows, three columns, two holes in each. To determine the number of holes the number of colums can be used first either to determine the total number of buttons or the number of holes in a row, but the final result is the same. Indicate order by parenthesis, thus: $4 \times (3 \times 2) = (4 \times 3) \times 2$.
C-9 Associativity	Associativity holds for the factors of one and zero.	Extend the number properties.	Are these sentence true or false? $(4 \times 2) \times 1 = 4 \times (2 \times 1)$ $(3 \times 0) \times 2 = 3 \times (0 \times 2)$ $(1 \times 0) \times 1 = 1 \times (0 \times 1)$ What numbers make this true? $(\Delta \times 1) \times 0 = \Delta \times (1 \times 0)$ Have students discuss patterns.
C-10 One	If one of the factors is one, the product is the same as the other factor.	Study the properties of one in multiplication.	Have the student interpret 5×1 using sets:  Ask for the product.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR THE STUDENT	PURPOSE AND OBJECTIVES	TEACHING EXAMPLES AND APPROACHES

Use many examples and compare the factors and the products:
$3 \times 1 = \Delta$, $7 \times 1 = \Delta$, $6 \times 1 = \Delta$, etc. $\Delta \times 1 = 3$, $\square \times 1 = 4$.
Ask the child if a product such as 5 can be interpreted in another way. (One set of 5, written as 1×5.)



Repeat with other examples and use frame arithmetic.

$1 \times \Delta = 6$; $1 \times 7 = \Delta$; $\Delta \times 9 = 9$; $1 \times \Delta = \Delta$.

Approach 2.
Another approach could be with repeated addition 1+1+1+1+1 could be expr[illegible]ed as 5×1. Then since the product is 5 and since 2×4 and 4×2 are two names for the same number, similarly $2 \times 3 = 3 \times 2$) $5 \times 1 = 1 \times 5$.

Approach 3.
Use patterns as developed from the matrix:

$2 \times 4 = 8$	$3 \times 3 = 9$
$2 \times 3 = 6$	$3 \times 2 = 6$
$2 \times 2 = 4$	$3 \times 1 = \Delta$
$2 \times 1 = \Delta$	

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	STEACHING APPROACH AND EXAMPLE
C-11 Zero	If one of the factors is zero the product is zero.	Study the properties of zero.	Use empty sets to illustrate $5\times0 = 0 + 0 + 0 + 0 + 0$ Ask for the product. Use many examples and compare the factors and the products. $5\times0=\triangle$; $4\times0=\square$; $2\times0=\triangle$ Ask the child if the product 0 can be interpreted in another way. (no sets of five written as 0×5)
C12 Distributive Property.	The distributive property offers a second way of solving the same problem. The distributive property links addition and multiplication.	Develop the number properties. (Distributive)	(a) $4+3=7$ (Number of boys plus number of girls equals number of children. $2x7=14$ (Number of pencils per child times number of children equals total number of pencils). (b) $2x4=8$ (Number of pencils per boy times number of boys equals the number of pencils distributed to the boys; $2x3=6$ (Number of pencils per girl times number of girls equals the number of pencils distributed to the girls). $8+6=14$ (Number of pencils distributed to boys plus number of pencils distributed to girls equals total number of pencils distributed. Summarize: $2x(4+3)= (2x4) + (2x3)$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLE
C-13 Multiplication with multi-digit multiplicands. (no carrying)	Multiplying numbers appearing as multi-digit numerals may be accomplished through repeated addition.	Extend the multiplication technique.	Think of 2 x 34 as 34 added two times. Work with one column at a time starting with the ones column. Think of as two times four ones and two times three tens. Write as: $\begin{array}{rr} \text{tens} & \text{ones} \\ 3 & 4 \\ \times & 2 \\ \hline 6 & 8 \end{array}$ From the two digit multiplicands, move into three and four digit multiplicands where the subproduct is nine or less. $\begin{array}{r} 133 \\ \times\ 2 \\ \hline \end{array}$ $\begin{array}{r} 3122 \\ \times\ \ 3 \\ \hline \end{array}$ Then include problems with zero as one of the digits: $\begin{array}{r} 302 \\ \times\ 3 \\ \hline \end{array}$ and $\begin{array}{r} 30 \\ \times 3 \\ \hline \end{array}$
C-14 Decade. multiplication. Single digit multiplier.	Four time twenty may be interpreted as (4x2)x10. The associative property is useful in interpreting multiplication.	Extend multiplication computational skills.	4 × 20 This combination is unfamiliar. Student must rewrite so combinations are single digits. Rename 20 and associate. 4 × (2x10) (4 × 2)x10 8x10 (familiar combination) 80 - simple name for 8 tens from study of place values.

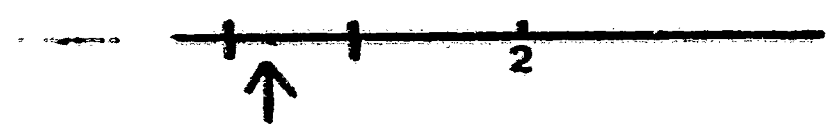
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
C-15 Two digit multiplicand. Single digit multiplier.	Multiplication with multi-digit multiplicands requires using the distributive property. Appreciate the power of number properties in extending computation	Extend multiplication computational skills. Appreciate the power of number properties in extending computation.	3×12 Student does not know this combination We must have smaller numbers. Rewrite 12 in a variety of ways: $3 \times (6+6)$ $3 \times (8+4)$ $3 \times (10+2)$ Apply the distributive property and substitute known products. $(3\times6) + (3\times6) = 18+18$ $(3\times8) + (3\times4) = 24+12$ $(3\times10) + (3\times2) - 30+6$ $\begin{array}{l} 12 \\ \underline{\times 3} \end{array} = \begin{array}{l} 1 \text{ ten} \\ \underline{\times 3} \\ 3 \text{ tens} \end{array} + \begin{array}{l} 2 \text{ ones} \\ \underline{\times 3} \\ 6 \text{ ones} \end{array}$
C-16 Division Relationship to set.	Division is related to the process of finding the number of equivalent subsets.	Readiness for division.	Take 6 books. Ask the students to separate into subsets of 2. How many equivalent subsets were formed?
C-17 Division Notation.	Symbolism is used as a short way of indicating division.	Develop communication skill.	Refer to C16. Write in various forms: $6 \div 2 = 3$ $\quad 2\overline{)6}^{\,3}$ $\quad \frac{6}{2} = 3$ Read all three as six divided into twos, (not as "two divided into six nor as "two ones into six.)

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE AND OBJECTIVES	TEACHING APPROACH AND EXAMPLES
D-18 Division relationship to subtraction.	Division may be interpreted as repeated subtraction. The quotient is the number of subtractions required.	Relate the operations. Expand vocabulary for communication.	In item D-15, as sets of two are separated, record the subtractions: $\begin{array}{r}6\\-2\\\hline 4\end{array}$ $\begin{array}{r}4\\-2\\\hline 2\end{array}$ $\begin{array}{r}2\\-2\\\hline 0\end{array}$ The number of times two was subtracted is the quotient.
D-19 Division	Division may be interpreted as seeking the number of rows or columns in an ordered array when the elements and the total are known.	Provide for multiple approaches and interpretations.	Ask a student to arrange 6 erasers in an array with 2 columns. The number of rows is the quotient. The numerical sentence can be expressed as [illegible]. Repeat but put in [illegible] rows and determine the number of columns.
D-20 Division Relation to multiplication.	Division and multiplication are inverse operation	Relate operations.	Illustrate 3x4. Divide the product by either factor. Note the relationship between the quotient and the other original factor. Repeat but divide first, then multiply. Discuss relationship between the operations as one undoing the other.

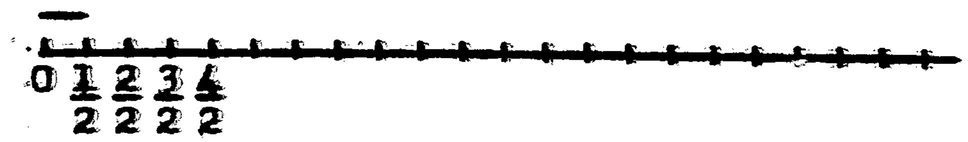
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
G-21 Definition of Division.	(a) Division is the operation on numbers by means of which the missing factor may be found when the product and one factor ~~are unknown~~. This operation is the inverse of multiplication. (c) Division is the operation on numbers by means of which a quotient and a remainder may be found whenever a whole number is divided by a counting number.	Define division precisely as an operation on number.	(a) Present problems such as $2 \times \square = 6$. Problem can be rewritten in terms of division as $6 \div 2 = \square$ or $2\overline{)\ 6}$ 3 is the number which makes the statement true: $2\overline{)6}$ with quotient 3 $6 \div 2 = 3$ because $2 \times 3 = 6$ (b) Present problem such as $2 \times \square = 7$ Rewrite as $7 \div 2 = \square$ or $2\overline{)\ 7}$ There is no whole number which will make the sentence true. 3 is the largest possible whole number factor but has a remainder and is written as $2\overline{)7}$ with 3 or 1 above similarly: $4\overline{)3}$ with 0 or 3 above
PROBLEM SOLVING G-22 Verbal problems	Certain situations can be expressed in terms of division.	Translate verbal problems to mathematical sentences and vice-versa.	A box of eight candy bars is distributed by giving two to each student. How many students receive candy bars? Express as: $2 \times \triangle = 8$ $8 \div 2 = \triangle$ Make up stories for: $3 \times \triangle = 6$

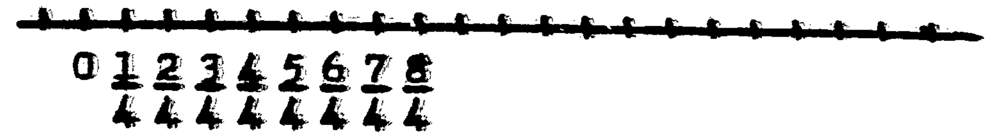
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR THE STUDENT	PURPOSES AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
NUMBER D-1 Addition. Multi-digit addends with carrying in more than one place.	Regrouping across columns makes possible addition with any number of digits in each addend.	To extend regrouping across the columns.	378 + 164 (300 + 70 + 8) + (100 + 60 + 4) (300 + 100) + (70 + 60) + (8 + 4) (400 + 130 + 12) (400 + 100) + (30 + 10) + 2 500 + 40 + 2 378 +164 12 8 + 4 130 70 + 60 (7 tens + 6 tens) 400 300 + 100 (3 hundreds + 1 hundred) 542 378 +164 Extend to larger numbers such as: 2,395,876 +1,946,895
D-2 Addition with more than two addends.	With three or more addends, the operation is: the first two addends are combined; their sum becomes the addend to be used with the 3rd addend.	To extend column addition. To appreciate that addition is a binary operation.	6) 9) 3) 13) 4 21) 8 23 7

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSES AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-3 Subtraction Regrouping	In order to complete some subtraction problems the numbers have to be renamed.	Enable computation in problems where subtraction in a given column is impossible until regrouping is performed.	$\begin{array}{r} 684 \\ \underline{-129} \end{array}$ can be renamed 6 hundreds, 7 tens, 14 ones − 1 hundred, 2 tens, 9 ones. $\begin{array}{r} 827 \\ \underline{-268} \end{array}$ can be renamed 7 hundreds, 11 tens, 17 ones − 2 hundreds, 6 tens, 8 ones
D-4 Closure	Only some whole numbers can be subtracted to obtain a difference of a whole number. (The set of whole numbers is not closed for subtraction.	Explore number properties. (closure)	Have the students and teacher make up ten problems for addition. Now ask the students to subtract them. (Be sure that in some cases the subtrahend is greater than the minuend.) Note that in the set of whole numbers there is not always an answer. $\begin{array}{r} 26 \\ \underline{+13} \\ 39 \end{array}$ $\begin{array}{r} 26 \\ \underline{-13} \\ 13 \end{array}$ $\begin{array}{r} 21 \\ \underline{+47} \\ 68 \end{array}$ $\begin{array}{r} 21 \\ \underline{-47} \end{array}$ The set of whole numbers are closed for addition because we can get a sum. Whole numbers are not closed for subtraction because there is not always a difference.
D-5 Commutative Property	Subtraction is not commutative.	Explore number properties.	Is this sentence always true? $\triangle - 5 = 5 - \triangle$ $\triangle - \square = \square - \triangle$ also try $\begin{array}{r} 8 \\ \underline{-5} \end{array}$ $\begin{array}{r} 5 \\ \underline{-8} \end{array}$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSES AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-6 Associative Property	Subtraction is not associative.	Explore number properties.	Ask students to solve the problem: 10-3-2= $\square$, then try 10-10-8= △ . Show that two different answers can be obtained depending upon interpretations, (10-3) - 2 = △ or 10-(3-2= △ and (10-10)-8 = △ or 10-(10-8) = △. The student is perfectly correct in interpreting the problem either way unless the parenthesis are included in the original problem. We are concluding that: 1. Parenthesis are needed to avoid ambiguity. 2. Since different grouping arrangements produce different results, subtraction is not associative. Are parenthesis needed in 10+3+2 = △ and why?
D-7 Fractions.	There are numbers other than whole numbers. They have a definite relationship of order to the whole numbers. Some fractions are larger than one.	Extend the number system; to study the number properties of order.	Draw a number line with 1 and 2 on it and ask what number would be represented by this point?  Would a number go here? If so, what? Repeat for 1/4. Try out 3/4. Ask whether a number goes between 1 and 2? Ask if they could find another name for 1,2, etc. using this notation. Use the numberline to have the following line completed:

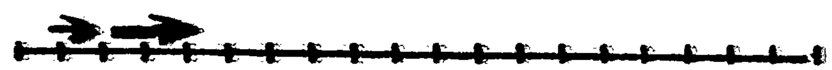
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSES AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES





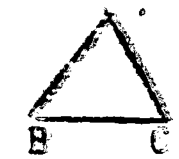
Repeat with thirds and sixths.
Note that whole numbers have fractional names and indicate which fractional numbers have whole names. Also note that one-half has other names such as $\frac{2}{4}$.

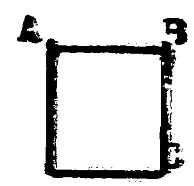
TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSES AND OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-8 Addition of fractions.	Fractions can be added.	Develop readiness for addition.	Use the number line to add $\frac{1}{4}$ and $\frac{2}{4}$



0 $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ to add $\frac{1}{2}$ and $\frac{1}{4}$ think of $\frac{1}{2}$ as $\frac{2}{4}$ and use the numberline.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-9 Verbal Problems	Fractions relate to parts of concrete objects. Certain verbal problems when represented as a mathematical sentence involve fractions.	Relate these new numbers to the real life situations.	Show representations of three halves and four halves with objects such as grapefruit, apples and flannel board objects. Use $\frac{1}{2}$ of a sandwich and $\frac{1}{2}$ of an apple and ask children if they can put the two halves together so that they will make a whole object. Repeat with examples where the addition would be meaningful. Tell the story for $\frac{3}{4} + \frac{2}{4} = \triangle$ Write a sentence for: John played for $\frac{1}{2}$ hour with Tim and 2 hours with Billy. How long did he play?
D-10 Fractions of groups.	Fractions relate to parts of groups. The group is considered as the unit.	Relate fractions to real life situations.	There were 18 boys and 14 girls in a class. Half the boys and half the girls were on the red spelling team. How many children were on the blue team?
D-11 GEOMETRY POINT	A point is an exact location.	Introduce and name points.	Show a cube. State the cube is made up of many points. Ask the students to indicate one of these points. (The child may point to any location on the surface and be correct) Raise the question as to whether the corner is a point. Let us give each of the corner points a name. Letters may be used.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-12 Line Segment	A line segment is straight and has a beginning point and an end point. These points can be named for purposes of communication.	Introduce the segment for use in measurement and name for communication purposes.	Review the properties of the edges of the cube. (The fact that it is straight and it has end points) This edge from corner to corner represent a line segment. Ask the children to draw a picture of this line segment and name the line segment. Repeat with a triangle.  The path from one corner to the next corner represents a line segment. Name the points at the corner of the triangle and then give names to the segments. The student is correct in using AB or BA.
D-13 Line	A line has no ends.	Distinguish between a line and a line segment.	Have the students draw a representation of a line segment. Name the end points. (For example K and N). Are there other points on the segment? Is it possible to make a line segment longer? How much longer? Can it go in both directions? Is there any end? Name some other points that were not on the original segment. This is a line. It is made of many points.  On a drawing arrows are used to show that the line continues and has no end.
D-14 Equality and Congruence.	Equality is used only for different names for the same figures. Congruence is used for different figures when a copy of one will exactly cover another.	Differentiate between equality and congruence.	Draw a square and ask for two names for the same segment.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHING APPROACHES AND EXAMPLES
			 AB = BA. The equality sign is used when different names for the same segment are being related. Is BC another name for AB? Could we therefore say BC = AB? No! Make a copy of BC by tracing or by marking the end points on the edge of another piece of paper. (Do not use a ruler) Compare the copy with AB to see if it covers AB exactly. If it does we may say AB and BC are congruent. This may be written as AB ≅ BC.
D-15 MEASUREMENT Introduction	A common unit of measure is necessary. The selection of the unit is arbitrary.	Develop a need for a standardized unit.	Ask the children to measure their desks, etc. using a unit (such as a hand, book, etc.) they choose. Let them compare their measurements of the same object with a neighbor. Discuss the necessity for a common unit of measure.
D-16 Linear Measurement.	Some common units of measurement are inch, foot, yard. The unit of measure used depends upon the length of the thing being measured.	Learn the common units of linear measure and to develop skill in usage.	Introduce by measuring various lengths such as the edge of a book the table, the floor to the nearest unit used, (such as about 8 in., about 4 ft., about 10 yards). (Use games).
D-17 Money	Measures in money can be added to or subtracted from other measures of money. Such measures can be multiplied or divided by whole numbers.	Extend concepts of money and develop using the four operations with dollars and cents. (Addition, subtraction, multiplication, division).	Computation with money. Use examples such as berbal problems involving addition, subtraction, multiplication, division.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVE	TEACHING APPROACHES AND EXAMPLES
D-18 Time	One minute is the basic unit for measurement on the clock. Another name for 2:15 is quarter past two; 2:30 is half-past two; 2:45 is quarter of three.	Teach the child to tell time to the minute. To learn other common names used in reading time.	Use the numberline showing zero to sixty and wrap it around the clock. Repeat B-36 reading to the nearest minute. Ask for other names for 2:15; 2:30; 2:45.
D-19 Temperature	Temperature can be recorded with negative numbers.	Develop readiness for negative numbers.	Read the temperature on a model thermometer that indicates below zero. Ask the students what the thermometer reading would be if the temperature was 3 degrees and dropped ten degrees. It would be recorded -7 degrees. What will be the reading be if it now rises 15 degrees?
D-20 Liquid Measure	A common unit of measure is necessary. The selection of the unit is arbitrary.	Appreciate the need for a standardized unit for measurement.	Use a large pan. Allow children to choose their own unit from a collection of different sized and shaped containers (baby food jars, peanut better and canning jars, bottles, etc.) Let each child completely fill the large container with water and record how many of his unit containers it takes. The number used is called the measure. If another room is doing this, using larger unit containers but the same size large pan, students will assume they have different size pans when the compare the measures. When they find out the pans are the same size the need for standard unit containers becomes apparent.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-21 Liquid Measure Common units.	Some common units are cup, pint, quart, half-gallon and gallon.	Learn the common units of liquid measure.	Have containers which represent these common units. Give them names and then develop the relationships by pouring water from one container to another. Record results in a table form.
D-22 Weight	The unit of weight does not have to be the same size, shape or composition as the object measured. The only common characteristic is the quality of weight. The selection of the unit is arbitrary.	Appreciate the basic need for a standard unit. To appreciate the unique quality of a unit of weight.	Bring in a rock. Take different objects (block of wood, book, pebbles, brick, paper weight, glass of water, sheet of paper, pencils, a set of standard weights). Could any of these serve as a basic unit of weight for the rock? Use the balance scale to determine the number of units of these objects necessary to put the two sides in balance. Discuss the desirability of a standard unit of weight. Compare pounds of different kinds of material.
D-23 MULTIPLICATION Basic combinations.	Patterns can be used to determine unknown combinations.	Review, to determine the combinations already known and extend to the remaining combinations. (up to 9 x 9).	Draw a blank matrix and have the students supply the products of known combinations. Ask students to find patterns to determine unknown products. Some of these ways will be: counting patterns (such as counting by 3's, 5's, etc.); commutative patterns; use of the distributive property. Examples: (1) $8 \times 7 = 8 \times (6 + 1) = (8 \times 6) + (8 \times 1)$ Use of a smaller combination already known (2) $7 \times 9 = (5 + 2) \times 9 = (5 \times 9) + (2 \times 9)$ (3) $8 \times 9 = 8 \times (10 - 1) = (8 \times 10) - (8 \times 1) = 80 - 8$

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACHES AND EXAMPLES
D-24 Multi-digit multiplicands with regrouping. (Single digit multiplier)	Multiplication of larger numbers by any single digit multiplier is made possible by regrouping.	Extend multiplication skills.	27 x 3

(1) $3 \times 27 = 3 \times (20+7)$ Using the distributive property

$= (3 \times 20) + (3 \times 7)$

$= (3 \times 20) + 21$

$= 60$ Use regrouping method previously

$= 81$ learned in addition

(2) Shorten to this form:

```
  27
 x 3
  21  = 3 x 7
+ 60  = 3 x 20
  81  = 3 x 27
```

(3) If the regrouping can be remembered it is not necessary to write the partial products:

```
 27
x 3
 81
```

We say: $3 \times 7 = 21$ ones
Exchange 20 ones for 2 tens
Then say: 3×2 tens are 6 tens. Six tens and the 2 tens I remember make 8 tens. The product is 81

```
 709
x  4
2836
```

We say: $4 \times 9 = 36$ ones
Exchange the 30 ones for 3 tens and write the 6 ones. Then we say 4×0 tens = 0 tens. This is shortened to: There are no tens. Write the 3 tens I remembered.
4×7 hundreds = 28 hundreds. 28 hundreds = 2 thousands 8 hundreds which we write.

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES
D-25 DIVISION WITH REMAINDER. (This might well be taught when D47 was introduced)	When finding the number of equivalent subsets of a given set we often have a subset which is not equivalent.	Extend the understanding of division.	Take a set of 7 objects and divide the set into equivalent subsets with 2 members each. We have 3 subsets with 2 members each and a subset of 1 left over, which we call the remainder.
D-26 Notation	Symbols are used in writing division.	Communicate ideas.	$7 \div 2 = 3$ twos R 1 This is read as: 7 divided into sets of two equal 3 sets of two with a remainder of 1. $2\overline{)7}$ with 3 R 1 above. Read as above. Equal Sign; Division sign $\frac{7}{2} = 3R1$ Read as above. When writing a division problem ALWAYS write the dividend first, the sign second, the divisor third, equals sign fourth.
D-27 Quotient not greater than 9, and divisors less than 9.	Division uses skills in addition, subtraction, and multiplication that have already been learned.	Develop in skill estimating the quotient.	How many sets of 4 do you think there are in 20? Ask the child, "Why do you think so?" Have him show you why he thinks so. Some children will use repeated subtraction, or repeated addition, or their knowledge of the multiplication fact involved. These are 3 different approaches and all are correct. Repeated subtraction: 20, −4, 16, −4, 12, −4, 8, −4, 4, −4, 0 Repeated addition: 0, +4, 4, +4, 8, +4, 12, +4, 16, +4, 20

TOPICAL OUTLINE	SPECIFIC UNDERSTANDING FOR STUDENTS	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	
D-28 Division by multidigit divisors with quotients not greater than 9.	Division uses skills in addition, subtraction, and multiplication that have already been learned.	Extend skills in estimating quotients.	$13\overline{)\,39}$	Ask how many sets of 13 do you think are in 39? Could there be as many as 4? Could there be as few as 2? Why do you think so? This approach can be extended to three digit divisors and dividends. Also use problems with remainders.

BASIC UNDERSTANDING AND SUGGESTED TECHNIQUES

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
WHOLE NUMBERS What is number? E-1 Set symbolism.	Develop ability to communicate through use of set symbols.	Make a vertical list of names of children with brown eyes, etc. Make horizontal list of these sets of names. Enclose the horizontal lists in brackets, braces or simple closed curves. For example: {Tom, Helen, Harry}. Read as the set of children with blond hair. This may be called "Set A". We would then write Set A = {Tom, Helen, Harry}.	There are standard symbols used in set notation. For example: { } are used to contain a set. [ ] and are symbols which are used to contain a set. They are not standard but sometimes substituted for ease of writing. Capital letters such as A, C, D are used to name sets.
E-2 Set basis for number.	Review ordering sets. Build readiness for transitivity.	Have students order a collection of non-equivalent sets e.g. D= {**}, A={*****}, E={*}. (See A-4) The illustrations of the sets can be made on cards and hung on a line. Pupils may have individual sets at their desks.	Sets can be ordered. If Set A has more elements than Set B and Set C has more elements than Set A then Set C has more elements than Set B.
E-3 Equivalent and non-equivalent sets.	Establish readiness for number classes.	Have students order collections of equivalent and non-equivalent sets. {**}, {////////}, {//}, {''''''''}	Equivalent sets occupy the same position in the order.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-4 Cardinal numbers.	Order cardinal numbers in readiness for the number line.	See A-5. Have the students select cards with the correct numeral and tag each set with its cardinal number name. (The numeral tag and the set, however, are not the same). If A = {0,*,#}, N (A) = 3 and is read: the cardinal number of Set A is three. (The set itself is not three).	The order of the sets determines the order of the cardinal number.
E-5 Number line.	Develop multiple approaches. Review and readiness for rational numbers.	Draw number lines, label several points and ask children what other points are on the number lines. (Have the child identify points beyond and between the points originally labeled). 3 9 15 23 30	Review and build one-to-one relationship between the numbers in arithmetic and the points on the number line. The whole numbers are a subset of the set of all numbers.
E-6 Empty set.	Review empty set. Introduce symbolism.	See A-21. Ask children to find a set for which zero is the cardinal number. Introduce symbolism.	{ } indicates empty set (no elements inside). ∅ is the symbol for the null or empty set.
E-7 Types of sets.	Broaden the concept of sets to include ideas which cannot be seen or handled.	Encourage students to think of sets whose elements are not concrete objects. (e.g. set of sounds, smells, even numbers, between twenty and thirty,etc.).	Sets can consist of concrete objects or abstract ideas.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-8 Finite sets.	Reinforce background for work with the set of whole numbers.	Ask the children to list finite and infinite sets e.g. the set of whole numbers less than five, the set of whole numbers between 15 and 25, the set of numbers which make true the sentence $\square + 3 = 3 + \square$ or $\square + 0 = \square$, the set of whole numbers greater than 10, etc.	Sets whose elements can be counted are finite.
Infinite sets.			Infinite sets are unending and therefore cannot be counted.
E-9 Symbolism for the infinite set.	To facilitate the listing of infinite sets.	The children will realize from E-8 that a method is needed to represent an unending set. The customary method is $\{1,2,3,\ldots\}$ or $\{2,4,8,26,32,\ldots\}$ and should be introduced only after children have tried their own methods of illustrating unending sets in E-8. See if children can continue the series in such examples as $\{1,\ldots\}$, $\{3,6\ldots\}$ to illustrate the need for showing enough numbers to clearly establish the pattern. The set $\{1,\ldots\}$ could be $\{1,2,3,\ldots\}$, $\{1,3,5,7,\ldots\}$, $\{1,3,9,27,\ldots\}$, $\{1,3,7,15,\ldots\}$ and so forth.	A way of listing the elements of the infinite set is to end the sequence with three dots. Enough elements should be shown in each set to clearly establish the number pattern.
E-10 Finite sets. (Different sizes)	Recognize that not all large sets are infinite. Introduce convenient symbolism for finite sets of many elements.	Discuss concrete sets such as the people in the U.S.A., the stars in the sky, grains of sand on the beach, hairs on the head, women Presidents of the U.S.A., the children in the class listed alphabetically, the numbers from one to one billion. Are these sets finite? (Yes, including the empty set). Can the elements in the above sets be conveniently listed? No, hence	Finite sets can have many elements or as few as no elements. The three dots in the notation show the continuing pattern and the last element is written after the three dots.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		we will have to use a symbolism to facilitate the listing. One symbolism that is useful for most of the above examples is {1,2,3,... 1,000,000,000}, {Ann, David, Fanny,...Zebulon}. Note that the last element of a finite set follows the three dots in the notation.	
E-11 The set of whole numbers.	Recognize the set of whole numbers. Develop readiness for operations with whole numbers.	Define the set of whole numbers. What is the largest whole number? Is there a larger one? The set starts with zero and each subsequent number is one greater, continuing without end. Review D-7 to reaffirm the existence of other sets of numbers.	The set of whole numbers is {0, 1, 2, ...}
OPERATIONS E-12 Addition.	Review and extend calculation.	Problems of this type: $\begin{array}{r} 324,781 \\ +\underline{585,019} \end{array}$ Child should be able to take a problem like 524+6203+25+42702 = $\square$ and construct a traditional vertical problem for adding. Place value should be stressed. The teacher should permit good students to add without copying in vertical form. Work up problems using these types of combinations: 1. No regrouping across columns. 2. Regrouping across columns. a-With a 0 digit in one of the places of the 1st addend. b-With a 0 digit in one of the places of the 2nd addend. 3. Combinations resulting in 0 in a given column in the sum.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-13 Subtraction.	Review calculations and vocabulary.	While one idea can be correctly expressed in many ways, be consistent using either (1) addends, missing addend and sum: or (2) minuend, subtrahend and difference. Do not mix the individual terms used in (1) with those of (2). Review, using examples such as: $\begin{array}{r}607\\ \underline{-325}\end{array}$ $\begin{array}{r}617\\ \underline{-325}\end{array}$ $617 - 315 = \square$ $617 - 315 = n$ $645 - 337 = \square$ $645 - 337 = n$	
E-14 Regrouping across more than 1 column.	Extend calculations.	Situation: 0 in minuend has on the right a digit which represents a smaller value than the corresponding digit in the subtrahend. Have students seek ways of computing: 1. $\begin{array}{r}301\\ \underline{-9}\end{array}$ 2. $\begin{array}{r}502\\ \underline{-86}\end{array}$ 3. $\begin{array}{r}10{,}001\\ \underline{-8{,}005}\end{array}$ 4. $\begin{array}{r}8{,}000\\ \underline{-1{,}070}\end{array}$ 5. $\begin{array}{r}8{,}000{,}000\\ \underline{-5{,}090}\end{array}$ Some possible alternate approaches are: Using Ex. 4 <u>Two regroupings</u> $\begin{array}{lll} 8{,}000 & & = (7 \times 1000) + (10 \times 100) \\ 8{,}000 & & = (7 \times 1000) + (\ 9 \times 100) + (10 \times 10) \\ 1{,}070 & & = \underline{(1 \times 1000) + \qquad\qquad (\ 7 \times 10)} \\ 8{,}000 & -\ 1{,}070 & = (6 \times 1000) + (9 \times 100) + (3 \times 10) \\ & & = 6{,}930 \end{array}$ <u>One regrouping</u>: $\begin{array}{lll} 8{,}000 & & = (7 \times 1000) + (100 \times 10) \\ 1{,}070 & & = \underline{(1 \times 1000) + (\ \ 7 \times 10)} \\ 8{,}000 & -\ 1{,}070 & = (6 \times 1000) + (\ 93 \times 10) \\ & & = 6{,}930 \end{array}$	It is useful to rename numbers in more than one way. It is sometimes necessary to regroup over more than one column.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		Two regroupings (different notation) $8{,}000 = 7000 + 1000 = 7000 + 900 + 100$ $1{,}070 = \underline{1000 + 0 + 70}$ $6000 + 900 + 30 = 6930$	
E-15 Multiplication Vocabulary.	Review	Review Language - Some may have learned multiplier and multiplicand. Others use first factor and second factor. Perhaps teachers should work with both for growth.	
E-16 Multiplication with a single digit multiplier.	Review, utilizing the difficult combinations of 7,8,9.	Review with stress on digits 7,8,9 as multiplicand and 7,8,9 as multiplier; i.e. $\begin{array}{r} 97 \\ \underline{\times 8} \end{array}$	
E-17 Multiples of 10.	To extend computational skill to multipliers (beginning with 10 and multiples of 10.)	1. $\begin{array}{r} 3 \\ \underline{\times 10} \end{array}$ $\begin{array}{r} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ \underline{+3} \\ 30 \end{array}$ Use addition methods such as at the left to build the generalization indicated in the student understanding. 2. $\begin{array}{r} 3 \\ \underline{\times 100} \end{array}$ The form illustrated in 2. shows that 100×3 means 3 added 100 times. This is inconvenient so we commute	When multiplying by ten, the digits in the multiplicand recur in the product one place to the left of their original position. When multiplying by one hundred, the digits recur two places to the left, etc.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		and then add 100 three times. ($100 \times 3 = 3 \times 100$). From this we note the generalization that when 3 is multiplied by 100, the 3 recurs in the product two places to the left. This generalization extends to higher multiples of 10.	
E-18 Exponential notation.	Reinforce place value. Enhance communication and develop background for exponential form of expanded notation.	Use generalization from E-17 to show $10 \times 10 = 100$ and $10 \times 10 \times 10 = 1000$. Introduce exponential notation. From E-17 use $10 \times 1 = 10$, $10 \times 100 = 1000$ and $1000 \times 1 = 1000$. $10 \times 10 \times 10$ or 1000 is written as 10^3 as a shortened form. The factor 10 in $(10 \times 10 \times 10)$ is written one time when expressed in exponential notation. Count the number of times the 10 is used as a factor. The counting number, called the exponent, is written above and to the right as in 10^3. Read 10^3 as "ten to the third power". The 10 in 10^3 is called the base. <u>Examples</u>: 100 is written 10^2 and read as "ten to the second power". 10,000 is written 10^4 and read as "ten to the fourth power". Have students express numerals of up to five digits in expanded notation and then exponential notation.	Numerals may be represented as sums of multiples (powers) of 10.
E-19 Exponential notation.	Expand exponential notation to cover all place values for whole numbers.	How can the place value always be determined from the next smaller place value? How can the value be determined from the next higher place value? Note the pattern in the following. What exponent for ten would be used to represent 1?	1 may be represented as 10^0.

Place value	10,000	1,000	100	10	1
Notation	10^4	10^3	10^2	10^1	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-20 Distributive property.	Develop most useful form for multiplication of two-digit multiplicands.	Have students verbalize a generalization for examples developed in C-12 such as: $2 \times (4 + 3) = (2 \times 4) + (2 \times 3)$ Would this sentence be true when other numbers are used? Any other number? What numbers can be reordered in adding or multiplying because of the number properties (particularly the commutative property), and have the sentence remain true? Since $2 \times 4 = 4 \times 2$ and $2 \times 3 = 3 \times 2$ We can say $2 \times (4 + 3) = (4 \times 2) + (3 \times 2)$ Since $2 \times (4 + 3)$ represents the product of 2 and $(4 + 3)$ or 7, we can also commute the order of these two numbers in multiplication without changing the value. Thus Since $2 \times (4 + 3) = (4 + 3) \times 2$ $(4 + 3) \times 2 = (4 \times 2) + (3 \times 2)$	The product of a sum and a third number can be determined by adding the products obtained by multiplying the third number times each of the addends in the original sum. The distributive property of multiplication over addition can be represented in the following form: $(8 + 2) \times 5 = (8 \times 5) + (2 \times 5)$
E-21 Multiplication. Multi-digit multiplier.	Extend multiplication to include multi-digit multipliers.	1. Review properties: Commutative -- see A-17, C-4 Associative -- see A-1[illegible], C-[illegible] 2. Work with factoring of such numbers as $30 = 10 \times 3$, $70 = 10 \times 7$, $600 = 100 \times 6$ (see C-6). One of the factors should be a multiple of 10. 3. Review multiplication by powers of 10 (see E-17). 4. Example: $\begin{array}{r} 45 \\ \times 23 \end{array}$ $23 \times 45 = (20 + 3) \times 45$	Multiplication involving larger numbers is made possible by regrouping such as by use of the distributive property.

By the distributive property of multiplication over addition we can rewrite this as $(20 \times 45) + (3 \times 45)$

By the commutative property of addition we can rewrite as $(3 \times 45) + (20 \times 45)$

We now have the first partial product
$135 + (20 \times 45)$

In order to arrive at a single digit multiplier for our second partial product we factor the multiplier 20 into 2×10. We now have $(2 \times 10) \times 45$

Using the associative principle of multiplication this becomes $2 \times (10 \times 45)$
$= 2 \times 450$
$= 900$ (second partial product)

Hence $23 \times 45 = (3 \times 45) + (20 \times 45)$
$= 135 + 900$
$= 1,035$

5. After the children have had sufficient experience with the form developed in step 4, shortcuts can be introduced.

```
   45
 × 23
  135 = 3 × 45
  900 = 20 × 45  (from step 4: (2 × 10) × 45 =
1,035  (by adding the two partial products)
```

The final zero in the second partial product (900) is sometimes dropped Note that there are several commutative properties, associative properties, and distributive properties, depending on the opeartion involved. Note which one is involved in steps 4 and 5.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-22 Multiplication.	Reinforce understanding previously developed and extend to more difficult problems.	Use examples such as: 8079 ×906 48474 7271100 7319574 $906 \times 8079 = (900 \times 8079) + (6 \times 8079)$ From the above we can see there will be only two partial products, namely 6×8079 900×8079 $900 \times 8079 = (9 \times 100) \times 8079$ $= 9 \times (100 \times 8079)$ $= 9 \times 807,900$ $= 7,271,100$	There will be as many partial products as there are non-zero digits in the multiplier.
E-23 Distributive property of division over addition.	Build structural basis for division of two-digit dividends.	Ask a student to divide a pile of crayons into sets of three. Repeat but ask for suggestions to speed up the process with another student assisting. Instead of dividing the whole group into three parts, they will probably separate the group into two smaller groups, one for each child, and then separate each group into sets of three. Using the corresponding mathematical operations and the cardinal numbers of the sets, students can generalize: $(12 + 21) \div 3 = (12 \div 3) + (21 \div 3)$ Is the statement true when other numbers are chosen? Regardless of the numbers chosen?	The quotient of a sum divided by a third number equals the sum of the quotients obtained by dividing each of the addends by the third number.
E-24 Division.	Extend skills to multi-digit quotients.	1. $69 \div 3 = (6 \text{ tens} + 9 \text{ ones}) \div 3$ $= (6 \text{ tens} \div 3) + (9 \text{ ones} \div 3)$ $= 20 + 3$ $= 23$	Several approaches to division involving multi-digit quotients are re-grouping, repeated subtraction, finding the missing factor, multiplying multiples of ten times the divisor.

2.

$$69 \div 3 = (60 + 9) \div 3$$
$$= (60 \div 3) + (9 \div 3)$$

Since $9 \div 3$ is familiar to the children let us now explore $60 \div 3$ as follows:

We need to find how many 3's in sixty.

a. To do this a child may use repeated subtraction ---e.g.

$$\begin{array}{r} 60 \\ \underline{-3} \\ 57 \\ \underline{-3} \\ \text{etc.} \end{array}$$

This will lead to shortcuts (involving multiples of the divisor)---e.g.

$$\begin{array}{rl} 60 & \\ \underline{-12} & (4 \times 3) \\ 43 & \\ \underline{-15} & (5 \times 3) \\ \text{etc.} & \end{array}$$

Many children will use multiples of ten times the divisor because it is easy to calculate ---e.g.

$$\begin{array}{rl} 60 & \\ \underline{-30} & (3 \times 10) \\ 30 & \\ \underline{-30} & (3 \times 10) \end{array}$$

b. Another child may suggest that multiplication is the inverse of division, so that $60 \div 3 = \square$ becomes $\square \times 3 = 60$. Since 20 makes the 2nd sentence true, it makes the first sentence $(60 \div 3 = \square)$ true.

To continue:

$$69 \div 3 = (60 \div 3) + (9 \div 3)$$
$$= 20 + 3$$
$$= 23$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

Three possible approaches are:

A.
```
3)69
  12 | 4
  57
  15 | 5
  42
  30 | 10
  12
  12 | 4
   0 | 23
```

B.
```
3)69
  30 | 10
  39
  30 | 10
   9
   9 | 3
   0 | 23
```

C.
```
    3
   20
3)69
  60
   9
   9
   0
```

E-25 Division.

Extend skills.

```
     89 R18
83)7405
   4150 | 50
   3255
   2490 | 30
    765
    581 | 7
    184
    166 | 2
     18
          89
```

1. In answer to the question "How many eighty-three's are there in 7,405?" A child might estimate 50 since he had reasoned that 10×83 is not enough and 100×83 are too many.
2. The child's second estimate might be "There may be thirty eighty-three's in the partial dividend 3,255."
3. Since ten eighty-three's are too many, try a smaller number such as seven.
4. Final estimate is two eighty-three's in the partial dividend, 184.
5. Since eighteen is less than the divisor, eighteen is the remainder.
6. As a final step the sum of the estimated quotients is written in the quotient's place with the remainder.
7. Approaches E-24 A or C may also be used.

The same approaches used in less difficult division problems may be used to solve more complex problems.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
E-26 Division. Vocabulary.	Sharpen meanings and appropriate uses of terms.	Discuss meaning and use of the term factor. Since factors are always associated with a number and in a manner that the number equals the product of the factors, confusion would arise if 3 were called a factor of 22 in the sentence $22 = (3 \times 7) + 1$ The use of the term is, therefore, restricted. The need for terms in the situation with non-zero remainders suggests the extended use for the terms quotient, divisor, and dividend.	Appropriate terms used in the mathematical relationships are: Dividend divided by divisor equals quotient plus remainder. Product divided by factor equals factor. The terms dividend, divisor, and quotient may be used in all division problems. The terms product and factor are used only in division problems with zero remainder.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
E-27 Division	Develop the generalization. Relate division to multiplication.	To develop the generalization start with a problem such as $6 \times 4 = 24$; from that we get the division equation: $24 \div 4 = 6$. If we let the letter <u>a</u> represent 24, the letter <u>b</u> represent 4 and <u>c</u> represent 6 we can rewrite the division as: $a \div b = c$. Extend and practice using different numbers such as: $\square \times 7 = 84$ $6 \times \square = 96$ $3 \times 13 = \square$ $144 \div \square = 6$ $\square \div 13 = 13$ $288 \div 9 = \square$ (Do not use problems with remainders other than zero.) One way to help the student realize the importance of the if and only if portion of the generalization is to present problems such as: $15 \div 3 \neq 7$ ($\neq$ means <u>does not</u> equal) Explain why.	The product divided by a factor equals the other factor if and only if that factor times the other factor equals the product. This is sometimes expressed as $a \div b = c$ if and only if $c \times b = a$
E-28 Division. Checking.	Develop independent verification skills.	Use examples such as: 1. $2546 \div 38 = 67$ $67 \times 38 = 2546$ 2. $24912 \div 41 = 607 \text{ R } 25$ $(607 \times 41) + 25 = 24912$	Since multiplication and division are inverse operations we make use of this knowledge to check division computation. 1. If $a \div b = c$ then $c \times b = a$ 2. To check a division problem which has a non-zero remainder, multiply the whole number quotient by the divisor and add the remainder.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
E-29 Division. Zero.	Develop an understanding of division when zero is used as one of the factors or the product.	Use examples such as: 1. If mother has no cookies, how many cookies could she give each of her five children? Write the corresponding mathematical sentence, using the cardinal number of the sets involved and the mathematical operation (division) that corresponds to the set operation. $0 \div 5 = 0$	1. If the product equals zero and one factor is a counting number, then the other factor is zero. In our statement $a \div b = c$, if a equals zero and b is a counting number then c equals 0 since $c \times b = a$.
		2. If $6 \div 0 = \square$, can you find a number which would make the following statement true: $\square \times 0 = 6$ (see C-11)	2. Division by zero is meaningless because zero times any factor will always equal zero. In the statement $a \div b = c$, if a does not equal zero then b cannot be zero since c times b must equal a but c time 0 equals 0.
		3. Ask what number can you find to make this sentence true: $\square \times 0 = 0$ Then what number makes this sentence true: $0 \div 0 = \square$ Since any number makes this statement true, this division results in no single value. Ask - "Have you seen any other division where the answer may be any number?" Develop the concept that any binary operation should have a unique result to be useful.	3. If the product is zero and one factor is zero, the other factor may be any number. In the statement $a \div b = c$, if a and b are both zero it is impossible to determine a unique value for c. Hence, the missing factor (quotient) is undefined.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	PROCEDURE OR APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
[illegible]-30 Estimation. Need.	To build a feeling for reasonableness of results. To build skills that are useful in long division.	Discuss examples such as (1) needing to know the number of bricks in a house (to the closest brick) to determine the cost of building a house, (2) needing to know the number of miles (to the closest mile) on a vacation trip by car to California to determine the expected cost, (3) buying items listed at $8.35, $1.98, $4.25, $3.89 and receiving a bill for $42.23. (Extremes such as in (1) and (2) should arouse student reaction that the degree of precision is not appropriate for estimation.)	There is a need for estimation. Estimation includes only the most meaningful or useful aspects of a number. Estimation makes possible rapid prediction and assures reasonableness of results.
[illegible]-31 Rounding numbers. Rounding to tens. Numbers ≤ 100	To build skill in rounding for estimation.	Have students name the decades between 0 and 100 (including 0 and 100). Take numbers which are less than 100 at random and determine which decade is "closest." In case of dispute, have student subtract the number from the decade or the decade from the number to determine which difference is less. Use a number line if necessary to help the student determine which decade is "closest." Include examples such as: 79 (round up) 42 (round down) 3 (round to 0) 96 (round to 100) 75 (round to either 70 or 80) 60 (already expressed to "closest" decade)	In rounding numbers to tens, the ten which is the "closest" (provides the smallest difference) is chosen. Numbers which are equally "close" to two numbers may be rounded to either.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
E-32 Rounding numbers. Rounding to hundreds. Numbers ≤ 1000	To build skill in rounding for division.	Use a procedure similar to E-31 but including 897 (number close to an even hundred) 549 (number far from an even hundred) 951 (number which in rounding changes the next place digit) 39 (number that round to zero) 730 (number expressed in decades) 350 (number that rounds to either of two choices) 1000 (number already expressed in hundreds)	In rounding numbers to hundreds, the hundred which is the "closest" is chosen.
E-33 Rounding numbers. Rounding to thousands. Numbers ≤ 1000	To build skill in rounding for division.	Use a procedure similar to E-31, including numbers such as: 8 (one digit number) 700 (even hundreds) 230 (even decades) 843 (non-zero digits in all places) 509 (zero digit in tens place) 78 (two digit number) 984 (numbers closest to 1000)	In rounding numbers to thousands, the thousand which is the "closest" is chosen.
E-34 Rounding numbers. Rounding to tens. Numbers > 100	Same as E-31.	Proceed as in E-31, including examples such as 1897 where digits in places higher than tens are affected.	Same as E-31.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
E-35 Rounding numbers. Rounding to hundreds. Numbers > 1000	Same as E-32.	Proceed as in E-31, using appropriate examples.	Same as E-32.
E-36 Division. Estimation.	Sharpen skills of estimation and develop reasonableness.	Before calculation is started, children should record their estimate of what the answer should be. In doing estimation, place value and multiples of ten are used. $37\overline{)8,209}$ 1. Round the dividend to 8,000 and the divisor to 40. 2. Try to answer the question: "How many 40's are there in 8,000?" 10×40 is much too small, $100 \times 40 = 4,000$ which is half 8,000 so 200×40 would be a reasonable estimation. 3. The estimation also gives you the number of digits in the quotient, with very few exceptions.	Estimation leads to efficient computation. The ability to estimate helps to avoid ridiculous answers.
PROBLEM SOLVING. E-37 Verbal Problems. Division.	Translate a practical problem situation into a	Have the class construct a word problem which requires division to provide the answer. For example:	It is possible to express verbal problems in mathematical sentences.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
	mathematical sentence and find the solution.	There are 558 desks in the school and 18 classrooms in the building. What is the number of desks in each room if each room has the same number of desks? All of the following sentences may be used to express the problem: $558 \div \square = 18$ $558 \div 18 = \square$ $18 \times \square = 558$ $\square \times 18 = 558$ The solution should be written in words: e.g. "The number of desks in each room is 31."	Problems which deal with partitioning of a set into equivalent subsets would require use of division in the mathematical sentence.
5-38 Verbal Problems. Multi-operation.	Build skill in verbal problems.	Use verbal problems which involve more than one operation. Include those where placeholders are likely to be included in the middle of the mathematical sentence. Have students explore and develop their own methods of solution including (1) trial and error, (2) trial and error directed by analysis of previous attempts, (3) reasoning. Ex. 1. Mother bought 3 dozen eggs and used 6 eggs while baking. How many eggs does mother now have? $(3 \times 12) - 6 = \square$ 2. John was paid 75 cents for mowing the lawn. He received 5 nickels and the rest in dimes. How many dimes did he receive? $(5 \times 5) + (10 \times \square) = 75$	More complicated mathematical sentences are needed to represent some stories. Finding the numbers which make some sentences true requires more than one operation and in a particular order.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		3. How many ways could John be paid 75 cents if he is paid in nickels and dimes? $(5 \times \Delta) + (10 \times \square) = 75$	
E- 39 Verbal Problems. Necessary and sufficient information.	Build skill in verbal problems.	Use problems without numbers and ask how the answers could be determined. Ex. 1 - Bill gives an apple to each of his parents and divides the rest among his brothers. How many does each brother receive? (No. of apples - 2) ÷ no. of brothers = no. each receives. Mix problems with insufficient information and excessive information, with others. Let students note which problems have insufficient information and what additional information is needed. Ex. 2 - How much did Mary pay for 5 lb. of sugar and 3 cans of peas at 23 cents a can? Ex. 3 - Ann has eight dozen lollipops which cost 11 cents a dozen. How many lollipops does she have?	The question asked in a problem indicates which numbers included in the problem are used in computing the answer or whether all the numbers needed are given.
E-40 Factors and Primes.	Define prime and composite numbers.	Review ordered arrays. See C-2. Ask the children to see in how many different ways they can arrange a set of elements in ordered arrays. e.g. A set of 6 can be arranged as:	A <u>prime number</u> is a whole number whose only two whole number factors are one and the number itself. All whole numbers greater than one, which are not prime, are called <u>composite numbers</u>.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

```
. . .   or   . .   or . . . . . .   or .
. . .        . .                       .
             . .                       .
                                       .
                                       .
                                       .
```

A set of 7 can be arranged only as

```
. . . . . . .   or .
                   .
                   .
                   .
                   .
                   .
                   .
```

They now list factors corresponding to their arrays: For the above examples:

3×2, 2×3, 1×6, 6×1, 7×1, 1×7

After many examples of this type the children should begin to see that some numbers have only two factors (in the set of whole numbers) while others have more than two. Children will enjoy constructing "Factor Trees".

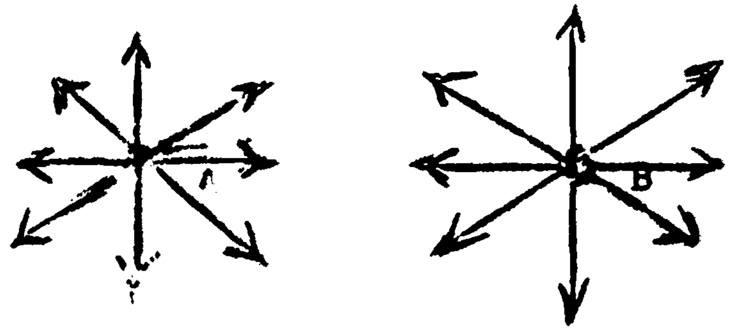
```
      24
 2 × 12
 2 × 2 × 6
 2 × 2 × 2 × 3
```

A device sometimes used to discover the prime numbers is the "Sieve of Erastosthenes".

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
E-41. Prime factorization.	Develop the ability to understand and use prime factors. Develop a concept that is useful for finding the greatest common factor.	1. Ask each student to factor 108. Factor in turn each new factor until only prime factors are contained in the indicated product. Note the various apprcaches used by different students. 108 / 36 × 3 / 6 × 6 × 3 / 2×3 × 2×3 × 3 108 / 9 × 12 / 9 × 3 × 4 / 3×3 × 3 × 2×2 108 / 2 × 54 / 2 × 6 × 9 / 2 × 2×3 × 3×3 The indicated products each contain the same prime factors and each prime number is used as a factor the same number of times. The result is unique except for the order in which they are written. The sequence of steps used in obtaining the product does not affect the result except for the order in which they are written. The product is often expressed as $2^2 \times 3^3$. 2. Another approach would be to use prime factorization and an idea similar to intersection. Intersection, however, only applies to sets. $36 = 2 \times 2 \times 3 \times 3$ $54 = 2 \times 3 \times 3 \times 3$ Construct the largest product which contains only factors common to both 36 and 54. Since 2 divides both 36 and 54, the largest common factor must contain 2 as a factor. Since 2×2 will not divide 54, the GCF may not contain 2 as a factor a second time. Since both 36 and 54 contain 3 as a factor twice, two three's would be contained in a common factor. Therefore, three would be contained in the greatest common factor. The GCF may contain no other prime factor since no prime factor divides both 36 and 54. $\boxed{2} \times 2 \times \boxed{3} \times \boxed{3}$ $\boxed{2} \times 3 \times \boxed{3} \times \boxed{3}$ $2 \times 3 \times 3 = GCF$	Every composite number has a unique factorization.

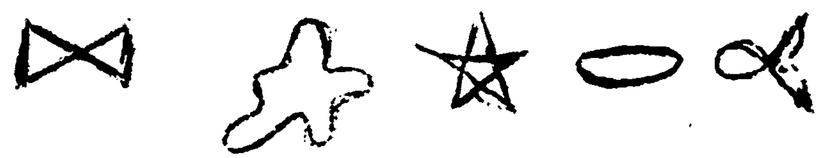
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
E-42 Divisibility in the decimal system.	Identify even numbers.	1. Give the children a set of numbers such as the following: 10, 3, 4, 8, 5, 14, 35, 26, 34, 19, 23. Ask the children to separate them into two distinct sets. The children are likely to separate these numbers in different ways: e.g. More than 10 and less than 10, odd and even numbers, prime and composite numbers, etc. Ask them to identify the rule they used to select their grouping. The rule for odd and even should be generalized out of this experience. Give additional practice, extending to larger numbers. Make up a set of large numbers and ask the children to use the generalization to identify the odd and even numbers. How?	1. The even numbers are the products of n × 2, where <u>n</u> is any whole number. <u>All</u> the even numbers are divisible by two (with no remainders). While authorities disagree, most include 0 in the set of even numbers.
	Identify numbers divisible by 5.	2. In experimenting with numbers to see which are divisible by 5, the children should develop their own generalization.	2. A number is divisible by 5 if and only if the last digit in its numeral is 0 or 5.
	Identify numbers divisible by 10.	3. See E-17	3. A number is divisible by 10 if and only if the last digit in its numeral is 0.
	Identify numbers divisible by 3.	4. Give children opportunity to try own experiments to discover the generalization. If the generalization does not come easily, then demonstrate it.	4. A number is divisible by 3 if the sum of the digits in its numeral is divisible by 3.
	Identify numbers divisible by 9.	5. See number (4) above.	5. A number is divisible by 9 if and only if the sum of the digits in its numeral is divisible by 9.
	Identify numbers divisible by 4.	6. See number (4) above.	6. A number is divisible by 4 if and only if the number represented by the last two digits is divisible by 4.
	Identify numbers divisible by 8.	7. See number (4) above.	7. A number is divisible by 8 if and only if the number represented by the last three digits is divisible by 8.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
GEOMETRY F-1 Points.	Introduce convenient representation of a point.	Point to various positions on the bare chalkboard. Make specific references to the unmarked points which would require the students to remember where they were such as: which point was highest, lowest, farthest apart, etc. Students should recognize the need for marking the points in some manner. Solicit possible ways to indicate points. (x, ., ¢, *) Discuss ways to name them. (Capital letters are most frequently used.) Discuss which is the best representation among the various size dots. What happens to the point represented by the top of a steeple when the steeple is removed? The point (position) does not move.	A dot is a representation of a point. (The smaller the dot, the better the representation.) A point represents a position.
F-2 Naming lines. Need.	Introduce names for lines. Develop readiness for line notation.	Review D-11. Have students draw many representations of lines on the board. Discuss these lines as to whether they are vertical, horizontal, crossing, etc. To avoid pointing students will refer to each line by the name of the person who drew it. Look for various other ways of referring to these lines such as letters, numbers, colors, etc.	To distinguish lines conveniently without pointing to them, names are needed.
F-3 Properties of lines.	Recognize that 2 points are necessary to deterimine a unique line.	Represent and name a point on the board or individually on paper. Have students make drawings of as many lines as possible through the point. Discuss how many lines were drawn, then how many could be drawn.	Many lines can pass through one point.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F. 4 Properties of lines.	Same as F-3.	Represent and name 2 points individually on paper. Again have students draw as many lines as possible through these 2 points. Review and discuss individual outcomes which may include:	Only <u>one</u> line can pass through 2 points.
		1. 	
		Student does not realize that the large dot is a poor representation of the points and false conclusions are being drawn about properties.	
		2. 	
		A perfectly correct interpretation but not the meaning intended.	
		3. 	
		The student interprets that the line must go through both points.	
		NOTE: The student should recognize that in the the future it is to be understood that the phrase "passing through 2 points" means that the same line must pass through both points. <u>Care</u> must be exercised in the wording of questions.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-5 Symbolism.	Develop precise symbolism and vocabulary.	Draw a line and represent 3 points on it such as:  How many ways can you name this line: Try to obtain AC, CA, AB, BA, BC, and CB. What sign can be used between any 2? (Ex. AC = CA, AC = AB, because all name the same line and include all the points on the line.) Name some segments from the same drawing. Does segment AB name the same set of points as line AB (Line AB includes all the points in segment AB plus C and all other points on the line whether named or not. Can we use the same symbol to represent line AB and segment AB? Introduce the new symbols $\overline{AB}$ (segment) and $\overleftrightarrow{AB}$ (line). Are $\overline{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BA}$, $\overline{CA}$, $\overleftrightarrow{BC}$ $\overline{CB}$ names for the same set of points? $\overleftrightarrow{AB} = \overleftrightarrow{BA}$, but $\overline{AB}$ does not equal $\overline{AC}$.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-6 Curve.	Introduce vocabulary.	Have students draw many paths between 2 given points. Indicate that these paths are sets of points. The general name for each of these sets of points is <u>curve</u>. Include the following:  The shortest curve, which includes the endpoints, has a special name called a line segment (sometimes shortened to segment). Also include some examples that cannot be conveniently represented on the board. Examples: paths of the astronauts, the space walk, path of a baseball pitch, etc.	A curve is a set of points. The shortest curve, which includes the endpoints, has a special name called a line segment (sometimes shortened to segment).
F-7 Plane.	Introduce vocabulary and meaning. of plane. Introduce Planar notion needed for common figures	Discuss models such as floor, wall, board, posters, desk top as <u>representatives</u> of sets of points. Emphasize that the models are flat and really should not end. Compare with a line which also does not end. Mark and discuss various subsets of points in the plane (such as selected points, segments, etc.)	A plane is a set of points. The plane consists of all the points represented by a flat surface, such as a wall or floor which did not end.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
	such as square or rectangle.		
F-8 Closed curve.	Introduce vocabulary and meaning of closed curve.	Have students suggest different paths which begin and end at the same point. Examples: path of runner who has hit a home run, orbit of a planet, etc. Have children illustrate examples on the board or map. Include examples such as: 	
F-9 Simple close curve.	Introduce vocabulary and meaning of simple closed curve and intersection.	From the previous examples, have children choose those curves no parts of which meet or cross. Indicate that this set of curves is given a special name called "simple closed curves." (only B and D are simple closed curves). Direct student attention to those closed curves which were not classed as simple. How do they differ? These points (where they cross) are examples of intersection. Define intersection.	A simple closed curve is the simplest form of closed curve, no part of which meets or crosses any other part. (An exception is recognized in that the beginning part and final part may be interpreted as meeting). The set of points or elements shared by two or more sets is called the intersection (the point where they meet or cross.)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-10 Intersection of sets.	Introduce the concept for future use in geometry and greatest common factor.	Have all the children in the first row raise their right hand and all the children in the first column raise their left hand. X ● ● ● ● ● O O O ● O O O The children will observe that one pupil belongs to both sets. Use the chalk board to show the same arrangment using names: Set A Set of children in first row = {Mary, James, Betty, Ann} Set B Set of children in first column = {Mary, Dick, Bob} We observe that Mary is a member of both Set A and Set B. $A \cap B = \{Mary\}$ This statement is read as: "The intersection of Set A with Set B is Mary". Using numerical examples: 1. $A = \{1,2,3,4\}$ $B = \{4,5,6\}$ $A \cap B = 4$	A set of elements shared by two or more sets is called the intersection of those sets. The symbol for intersection is $\cap$.

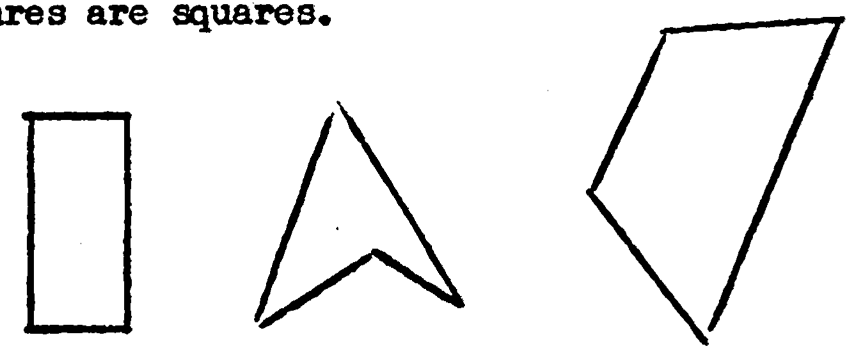
TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		2. $A = \{3, 9, 7, 10\}$ $B = \{7, 10, 15, 18, 21\}$ $C = \{10, 2, 9, 18, 7, 24\}$ $A \cap B = \{7, 10\}$ $B \cap C = \{7, 10\}$ $A \cap C = \{7, 9, 10\}$ $A \cap B \cap C = \{7, 10\}$	
WHOLE NUMBERS F-11 Greatest common factor.	Develop the ability to find the G.C.F. for use in renaming fractions.	1. Ask the children to list all of the whole number factors of 36 and 54: Set of counting number factors of $36 = \{1,2,3,4,6,9,12,18,36\}$ Set of counting number factors of $54 = \{1,2,3,6,9,18,27,54\}$ We observe that the common factors of these two numbers are $\{1,2,3,6,9,18\}$ The greatest of these common factors is 18. Note that the GCF does not have to be prime. Could 108 be a common factor of 36 and 54? No, not in this reference, because the other factor would be a fraction and the GCF is limited to counting numbers. Why? Otherwise, there would never be a <u>greatest</u> common factor.	The greatest common factor of two or more numbers is the largest number, which is a factor of each. Greatest factor refers only to numbers which are counting numbers.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		2. Another approach would be to use prime factorization. $36 = 2 \times 2 \times 3 \times 3$ $54 = 2 \times 3 \times 3 \times 3$ Since one 2 and two 3's are common to both factorizations and hence factors of both, then their product, 18, is a common factor. Since one 2 and two 3's are the only primes common to both, their product is the largest common factor.	
F-12 Union of Sets.	Introduce the symbol ∪. Facilitate definitions of geometric figures.	Use examples such as the following to demonstrate the endproduct when joining sets: A = {*, $, %} B = {o, *, #} A ∪ B = {*, o, $, #, %} X = {boys in this class} Y = {girls in this class} X ∪ Y = {children in this class} R = {children older than nine years} S = {children younger than eleven years} R ∪ S = {all children}	1. The union of two or more sets is the set which contains all of the elements which appear in any of the sets, without duplication. 2. A∪B indicates the union of Set A with Set B.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-13 Least common multiple.	Prepare for adding fractions. Develop ability to determine the least common multiple.	1. Have students note that the product of 9 and 15 is a multiple of each number but not the least common multiple. The product of 2 or more numbers is the least common multiple of these numbers only if they have no common factors. 2. Ask the children to list counting number multiples of 9 and 15. Multiples of 9 = {9, 18, 27, 36, 45, 54, 63, 72, 81, 90...135...} Multiples of 15 = {15, 30, 45, 60, 75, 90, ...135...} Have them find some common multiples of these two numbers and then ask them to identify the <u>Least</u> Common Multiple of 9 and 15. The L.C.M. Of 9 and 15 is 45. What is the greatest common multiple? (There is none) 3. Another approach to find the L.C.M. depends on the use of prime factorization. $9 = 3 \times 3$ $15 = 3 \times 5$ The L.C.M. of 9 and 15 is the product of $3 \times 3 \times 5$. Observe that the factor 3 appears twice in the prime factorization of 9 and only once in the prime factorization of 15.	The least common multiple is the smallest counting number which is divisible (with a remainder of zero) by each of the numbers for which it is the least common multiple.

TOPICAL OUTLINE	PURPOSE OR OUTLINE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		Every whole number multiple of 9 must be a multiple of 3×3. Therefore, every common multiple of 9 and another number must be a multiple of 3×3. Every whole number multiple of 15 must be a multiple of 3 and 5. 3×3 is already a multiple of 3 but not 5. Therefore, 5 is included as a factor. Since $3 \times 3 \times 5$ contains only factors absolutely needed, it is the least common multiple. No factor is needed more often than it occurs in the prime factorization of any number for which it is the multiple. 4. $140 = 2 \times 2 \times 5 \times 7$ $180 = 2 \times 2 \times 3 \times 3 \times 5$ $300 = 2 \times 2 \times 3 \times 5 \times 5$ $LCM = 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7$ $LCM = 6300$ To help the children check that all common factors in these three numbers have been used and that 6,300 is the L.C.M., the following technique is useful: Divide the numbers considered to be the L.C.M. by each of the original numbers and then factor each quotient into primes. $6{,}300 \div 140 = 45$ $45 = 3 \times 3 \times 5$ $6{,}300 \div 180 = 35$ $35 = 5 \times 7$ $6{,}300 \div 30 = 21$ $21 = 3 \times 7$ Note that the quotients, 45, 35, and 21 have no factors in common to all three. Therefore 6,300 is the Least Common Multiple.	
F-GEOMETRY F-14 Common simple closed curve.	Introduce common simple closed curves. Become aware of distinguishing	Review Union as in A-10 and F-12. Have students represent some common simple closed curves for which they have specific names. Discuss the characteristics which determine to which figures the names apply. Follow thru on students' suggestions to determine whether the characteristics are sufficient or necessary.	Triangles, quadrilaterals, polygons, rectangles, squares, and circles are examples of simple closed curves which are contained in a plane.

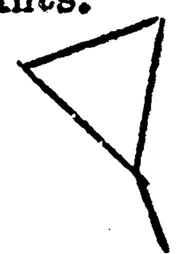
TOPICAL OUTLINE	PURPOSE OF OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
	characteristics of each. Learn how to classify knowledge accurately and concisely and learn how to communicate these ideas.	Example: A definition which says a square has 4 sides is countered with the illustration of a rectangle. Ask whether the following 4-sided figures are squares.	At this time we can only define some of these figures. A triangle is a simple closed curve formed by the union of exactly 3 line segments. A quadrilatoral is a simple closed curve formed by the union of exactly 4 line segments. A polygon is a simple closed curve formed by the union of 3 or more line segments.



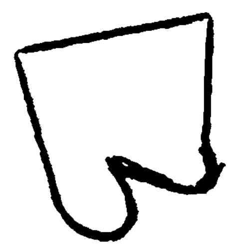
Develop definitions. (Refer to student understanding.) Include discussion of characteristics such as:

1.) All of these are simple closed curves.
2.) They consist of line segments (except circles).
3.) Union implies the joining of segments indicated and includes no other sets of points.

1.



2.



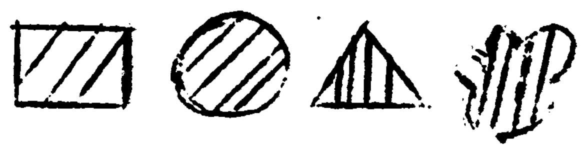
(These figures have 3 sides, but contain additional points.)

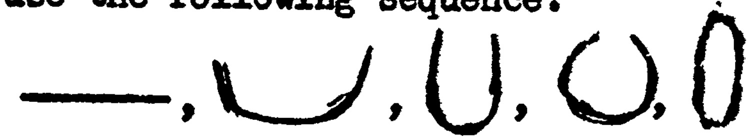
Definitions of terms such as square, rectangle, and circle depend on concepts of measure that have not yet been sufficiently developed. (Square and rectangle are not defined here because they depend upon the measure of angle for precise definition. Definition of circle depends upon the concept of congruent segments)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		Does (1) represent the union of 3 segments? Yes. Is it a simple closed curve? No. Figure (1) is therefore not a triangle. Does (2) represent the union of 3 segments? No. It has additional points. Is it a simple closed curve? Yes. Figure (2) is not a triangle. Discuss the relationship between various figures, such as: 1.) All triangles are polygons. 2.) A triangle cannot be a quadrilateral. 3.) Squares and rectangles are specific cases of quadrilaterals. 4.) A circle is not a polygon. 5.) Some polygons are squares.	
F-15 Infinite sets.	Build concept of infinite nature of most geometric figures. Reinforce concept of infinite set. Reinforce concept that a point has no dimension. Build concept that between any two points, there are additional points.	Review how many numbers are in the set of whole numbers. Review that each whole number is associated with a point on the number line. How many points are on the number line? A line is an example (non-numerical) of an infinite set. Ask student to draw a representation of a segment. Discuss the number of points on the segment. If a number is suggested, suggest adding an additional point between two others. Repeat if a larger number is suggested until student realizes there is no end. Is the segment a finite or infinite set of points? (infinite) Discuss whether squares, triangles, and the simple closed curves in general are finite or infinite sets	Lines, line segments, closed curves, and curves in general are infinite sets of points.

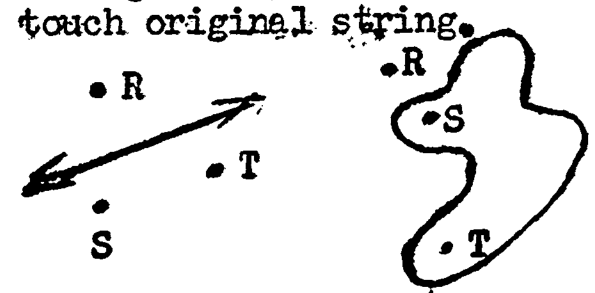
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		of points. (They are infinite sets.) Caution: Do not let student consider interiors in the discussion since the figure consists only of the border. Wires or strings may be used to make physical models.	
F-16 Separations. (2 dimensions)	Develop readiness for regions and area.	Ask children to determine how they would separate branded and unbranded cattle in an open field. The fence would take the form of what geometric figure? Would a 10 foot segment be sufficient? (No) Would a fence in the form of a line be satisfactory? (Yes, lines are unending) Can a fence in the form of a simple closed curve separate the cattle? Have each child use a sheet of paper to represent the field and illustrate their own ideas using several possibilities. Discuss outcomes -- use chalkboard illustrations and geometric figures as sets of points. How many sets of points are formed by these drawings? (Lines and closed curves form 3 sets counting the border itself as a separate set. A segment forms 2 sets, one of which is the segment.) Indicate that we call each of the sides of the plane separated by the line a half plane.	The points in a plane can be divided into separate and distinct sets. A line or any simple closed curve will divide the plane into 3 sets of points. A point or segment will separate the plane into 2 sets of points. A half plane is the set of points in a plane on one of the "sides" of a line. A half plane is one of the two or more sets of points into which a plane is separated (but excluding the boundary as one of the possible sets).

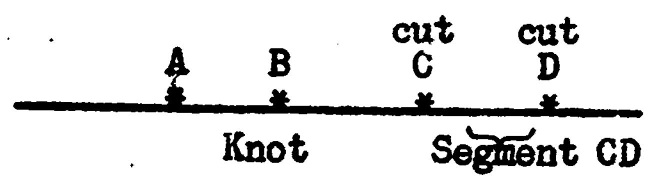
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-17 Space.	Develop basis for 3 dimensional geometry.	Ask students what is meant by space. How far does space extend? Is there space in the room? Are we a part of it? Does space stop at the earth's surface? Is there space in an oil well? Is there space in the core of the earth even if it is occupied? Give letter names to points in space including some points in the obscure locations in the examples above.	Space is the set of all points. Space may or may not be occupied by material objects. We can name any specific point in space.
F-18 Simple closed surfaces.	Introduce three-dimensional figures. Build the concept of two-dimensional properties (area) of three dimensional figures.	Use basketballs, footballs, boxes, cans, jars and other "hollow" models to demonstrate simple closed surfaces. Discuss possible definitions. Discuss the need to be able to "surround" a part of space and separate this part from the rest of space. Any curve that connects a point in the enclosed space to one outside the enclosed space must intersect the closed surface. Discuss possible intersections.	A closed surface is a continuous set of points in space which separates a portion of space from all other points in space. A plane is an example which is not a closed figure but which also separates one portion of space from all other points in space.

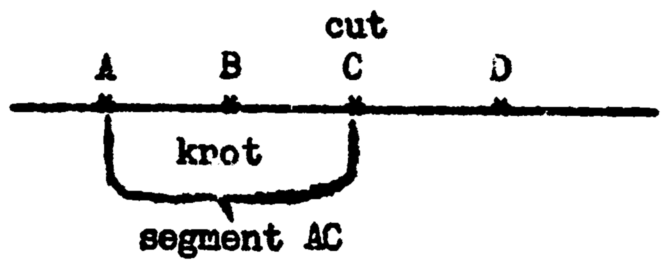
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-19 Separations. (Extension into 3 dimensions)	Develop readiness for space regions and volume.	Use the techniques in step F-16 but utilize 3 dimensional ideas. <u>Example</u>. Separate starlings from all other birds. What would you need to do this? (A cage, other closed surfaces, walls or planes that do not end, could be used). Introduce vocabulary of half space.	The points in space can be separated into distinct sets. A plane or any simple closed surface will separate the plane into 3 sets of points. A point, curve, or nonclosed 2 dimensional region will separate space into 2 sets of points. A half-space is one of the set of points into which space is separated by a plane, excluding the plane itself.
F-20 Regions.	Introduce vocabulary of region.	Use illustrations from F-16 and F-19 to define region. Use figures such as balls, balloons, boxes and construction paper to demonstrate. Sample regions include: 	A region is a geometric figure together with one of the set of points that it separates. Specific useful regions are those bounded by (1) a simple closed curve, (2) a simple closed surface.

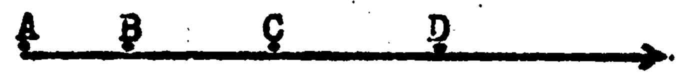
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		Discuss the difference between a wire model of just the boundary and a region of the same shape shown by a model made of construction paper. Hollow models such as balloons and tires can demonstrate surfaces and and examples such as solid wood models or doughnuts can illustrate regions.	
F-21 One-Dimensional Figures.	Build distinction between perimeter and area.	Indicate that string can be used to illustrate any one-dimensional figure. Have the student make as many figures as he can devise. If the student has not used the simple closed curve as an illustration, use the following sequence:  Be sure that the student recognizes that the string has not gained the dimension called "width". Are there geometric figures or ideas that cannot be represented by a string of any length? (Review concept that a string is only a representation of a curve and should have no thinkness.) Example: A region cannot be represented by string and therefore cannot be one-dimensional.	Any curve, closed or open, is a one dimensional figure. The region formed by a closed curve is not one dimensional.

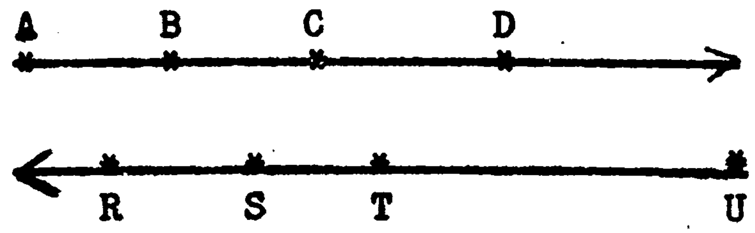
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	CHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-22 Separation of One-Dimensional Figures.	To refine the idea of separation.	Take a string and ask the students how they would separate it. The students will probably use an object that will represent a line (scissors, knife, etc.) Lead students to note that these tools represent a line or line segment. Could the segment be shorter? How short? Only a point is needed.)	Points are needed to separate one-dimensional figures.
		Take a simple closed curve and again ask how they would separate it. Two points are needed.	
		Seek names from the students for the figures formed by the separation of a line into parts by a point. Present the name, half line, as a possibility if students do not. Indicate this is the generally accepted term. A half line does not contain an endpoint. Obtain student reactions to the half lines obtained in the model: 	All points in a line which are on one side of the boundary (point) form a figure called a half-line.
		Since lines have no ends, the illusion that half lines are not really half is misleading.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
F-23 Location of points in relation to a line boundary. Location of points in relation to a simple closed curve.	Develop readiness for interior and exterior.	Have a piece of string representing a line placed on the floor from the front wall to the back. Establish and name any 2 points on the floor. Can you place another piece of string connecting the 2 points without crossing the original string? Lead the children to generalize that if 2 points can be connected by a curve that does not cross the line, they are on one side of the line. What can be said about the 2 points if the curve must cross the line? They are not both on the same side. Repeat with simple closed curve instead of a line. CAUTION: This generalization does not hold true if students attempt to utilize all the space instead of the plane. Example: if student lifts the string from plane so that it does not touch original string.  S and T can be connected without crossing the boundary. Therefore, S and T are in the same set (on the same side of the boundary). R and T cannot be connected without crossing the boundary. Therefore R and T are in different sets (on opposite sides of the boundary.)	When we speak of separation, we are restricted to utilizing points in the original set of points. Two points are on the same side of a line in the plane if they can be connected with a curve that does not intersect the line. Two points in the plane are on the same "side" of a simple closed curve if they can be connected with a curve that does not intersect the curve. Two points in the plane that cannot be connected with a curve unless the curve intersects the boundary are not both on the same side of the boundary.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
F-24 Location of Points in Relation to a Plane Boundary or a Simple Closed Surface.	Develop readiness for interior and exterior.	Have 2 students represent points A and B. Locate the "points" on opposite sides of a wall. Stress that the wall represents a plane. Can the points be connected by a string representing a curve without passing through the wall? (Plane — walls and doors are a part of that plane). How must the 2 points be relocated so that they <u>can</u> be connected by the string? Repeat procedure using closed surfaces {balloons (round, oblong), football, inner tubes}	Two points in space are on the same side of plane if and only if they can be connected by a curve that does not intersect the plane. Two points are on the same side of a simple closed surface if and only if they can be connected by a curve that does not intersect the surface.
F-25 Separations in unclosed curves.	Extend concepts of separation to one dimensional figures.	Put a knot in a piece of string and use it as the boundary (point.) Have a student designate two other points on the string. Cut the string at those points. Give the piece (CD) to another student.  What can be said about the two points where the string was cut if the string (CD) does not contain the knot? (Points are on the same side of the knot)	Two points on a curve that can be connected by a portion of that curve which does not intersect the boundary (point) are on the same side of the boundary. Two points on a curve that can be connected by a portion of that curve which does intersect a boundary (point) are on opposite sides of the boundary.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		 What can be said about the points if the string ($\overline{AC}$) does contain the knot? (Points on opposite sides of the knot)	
F-26. Interior and Exterior.	Prepare for understanding of area and volume.	Take various 2 and 3 dimensional figures. Discuss the need for specific names for the sets of points formed by the separation. These are arbitrarily assigned the names "interior", "exterior", and boundary. Discuss intuitively which is the interior and which is exterior.	Interior and exterior are the names of the 2 sides of simple closed curves or simple closed surfaces. The boundary is not included in either. The interior is arbitrarily assigned to the set of points that is "enclosed".
F-27 Rays.	Build a basis for defining angle.	Ask the students if there are other geometric figures other than those previously described. Suggest the set of points represented by the beam of a flashlight. Ask for suggestions for a name for the set. <u>Ray</u> is the common name. Discuss the difference between it and half line.	A ray is a straight geometric figure which is a continuous set of points with one endpoint. A ray includes a half line and the endpoint.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
F-28 Naming Rays.	Build a basis for naming rays.	Represent a ray which extends indefinitely to the right. Label the endpoint and several other points. Ask for suggestions for naming the ray. (AB, AC, AD)  Name two points for the student. Have him represent as many different rays as these 2 points suggest. Indicate that there is a need for naming a single set of points. How can this be done? Lead to the conclusion that the endpoint must be named and one other point to indicate direction. Indicate that it is necessary to know which of the 2 points is the endpoint. This is commonly done by naming the endpoint first. Ask for a suggestion for a symbol which will eliminate the need for writing the word ray, but does distinguish if from a segment or line. (Accepted symbol is $\overrightarrow{AB}$.)	A ray is named by its endpoint and one other point on the ray. It is symbolized by $\overrightarrow{AB}$ where the first letter is the endpoint.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		using this new concept, how many ways can you name these rays?  ($\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, are correct for the first. ($\overrightarrow{UT}$, $\overrightarrow{US}$, $\overrightarrow{UR}$, are correct for the second. Note again that the <u>A</u> endpoint is written first. Note also that if a point other than the endpoint is named first, we are naming a different set of points. The arrow in the symbol for ray never points to the left. Equal signs may be used between names for the same ray and only when both name the same ray. Ex. $\overrightarrow{AB} = \overrightarrow{AC}$ Also use rays in positions such as vertical for discussing rays. (Note: see F-5 -- Line Symbolism)	
F-29 Angles. (definition)	Build a basis for measuring direction and defining figures.	Represent some angles on the board. Discuss some properties; such as, the angle continues indefinitely and is composed of 2 rays. 	An angle is the union of 2 rays which do not form a straight line but have a common endpoint.

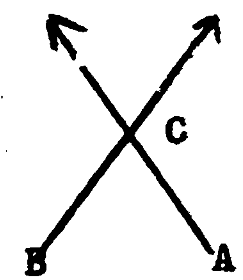
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

Could it be any 2 rays:
Example:

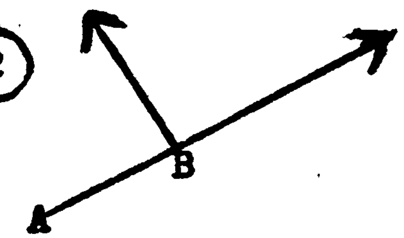
Conclusion: Rays must meet.

Do any 2 rays that meet form an angle: Example:

1



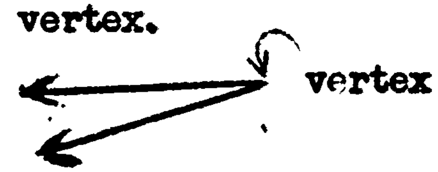
2



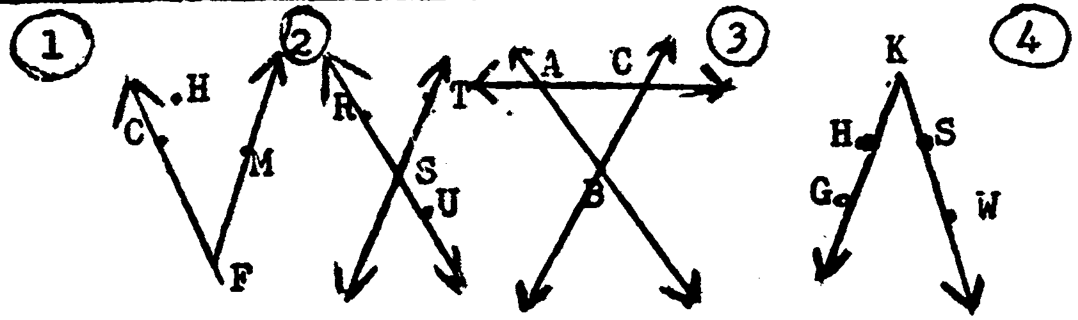
No! They must have a common endpoint.

The complete figures above (including the 2 rays whose endpoints are A and B) are not angles. Part of the figure, however, does form an angle. (for example the two rays in figure 1 that have endpoint C and the two rays in figure 2 that have endpoint B).
Unless otherwise indicated the figure is considered in its entirety for discussion purposes.
Do any 2 rays with a common endpoint form an angle? Example:

No!

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		The rays shown in the previous example form a line. Try to use some models to reinforce the concept that the angle contains <u>no</u> points except the rays. Example: hands of a clock, path of a ball that has bounced off the floor or wall, spokes on a wheel.	
F-30 Vertex. (definition)	Make naming of angles clearer.	Review intersections of other geometric figures. Ask what is the intersection of rays that form the angle and where it is. It is a point. This point is commonly defined as the vertex. 	The vertex is a point which is the common point of the 2 rays that form an angle. The common endpoint (vertex) is the intersection of these 2 rays.
F-31 Naming Angles.	Facilitate communication.	Have students construct an angle by representing and naming the rays involved. Ask for suggestions on how to name the angles. Find as many names for the angle as possible. Is point H in Ex. 1 included in the angle? No! Remember that the angle includes only the 2 rays and not the interior.	Angles are usually named by the vertex and an additional point on each ray with the common endpoint written between the two.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

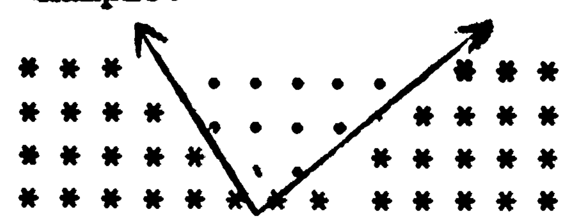


Is example 2 a representation of an angle? Does it include an angle? When we name an angle, we shall mean only the points included in the angle, regardless of whether or not other points are indicated. Which of the following are included in angle RST?

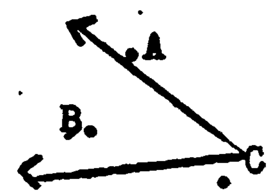
R, S, U, $\overrightarrow{ST}$, $\overleftrightarrow{SU}$? (only R, S, and $\overrightarrow{ST}$)

In example 3, could angle ABC be interpreted as many different figures? What must be known to limit it to one figure? (which letter represents the vertex). If B is the vertex in each case, do angle ABC and CBA name the same set of points? Do ABC and BAC name the same figure, if the central letter is always the vertex? No! To avoid confusion as to the figure involved, the position of the vertex must be known. The central letter is most commonly used to denote the vertex. Only one figure results.

The symbol $\angle$ is used for the word angle to shorten the writing of angle names

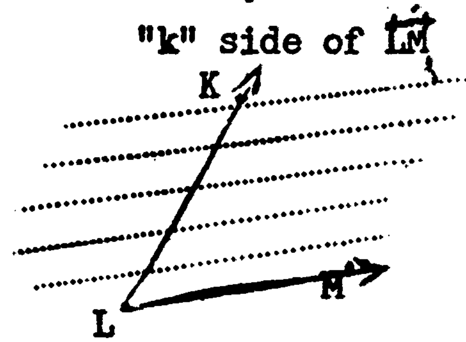
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
		as in $\angle$ ABC. (read as angle ABC). In example 4, list many possible names for the angle indicated. Example -- $\angle$ GKW, $\angle$ HKS, $\angle$ HKW. Since they are names for the same angle, we may use between them: $\angle$ GKW = $\angle$ HKS $\angle$ GKW = $\angle$ HKW	
F-32 Angles Separations	Develop readiness for angular regions.	Review methods of separating a plane into separate sets of points. Does an angle do the same thing? Point to the sets of points formed by the angle. (The angle itself is a distinct set of points which separates the plane into 2 other sets of points, the set of dots and the set of asterisks, yielding 3 distinct sets of points.) Example: 	An angle separates a plane into 3 sets of points. The angle is the boundary. Two points are on the same side of an angle if and only if they can be connected by a curve that does not intersect the boundary.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
		Does the previous generalization for determining whether 2 points are on the same or opposite sides still hold? Is there a better way for determining whether the following points are on the same or opposite sides of the boundary?	



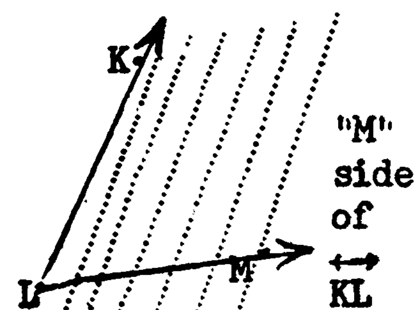
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
F-33 Angles. Interior.	Build a basis for measuring angular regions.	Represent angle KLM on the board. Discuss intuitively which set of points formed by the separation should be called the interior. Look for ways that define the interior. Ultimately lead to the definition. The following sequence might indicate the development.	The interior of angle KLM is a set of points formed by the intersection of the half plane on the K side of line LM and the half plane on the M side of line KL.

A.



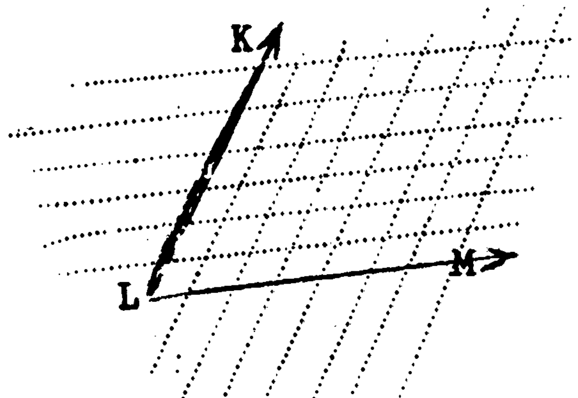
K and L determine a line (not shown)

B.



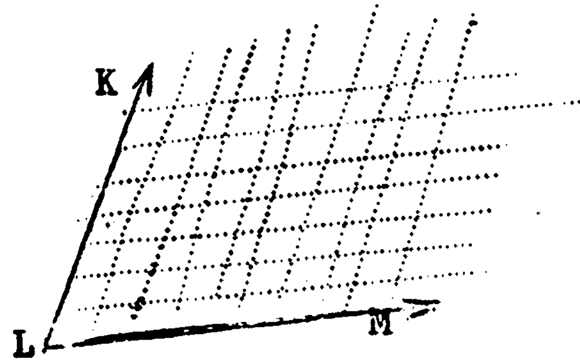
L and M determine a line (not shown)

C.



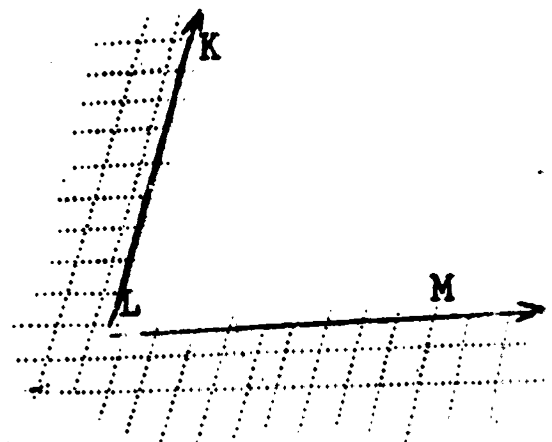
$\angle$KLM with its interior and exterior

D.



$\angle$KLM with its interior

E.



$\angle$KLM with its exterior

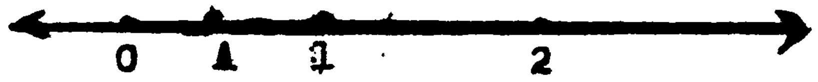
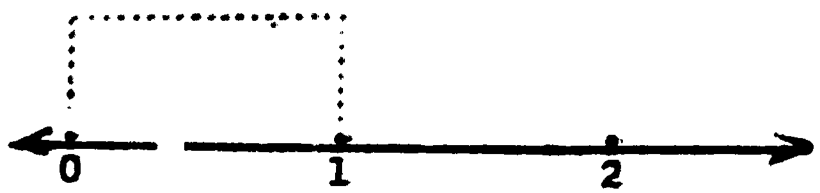
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENTS
		A better visual representation results by using colored acetate or cellophane to represent the half planes which are placed over each other to show the intersection. An overhead projector is useful for teacher demonstration. The other set of points formed by the separation is called the exterior. Have students mark several points in the exterior. Repeat with angles such as:  The hands of a clock can be used to represent many different angles and angles in many positions.	
F-34 Angular Region.	Build a basis for measuring angular region.	Review the fact that the interior does not include the boundary. A new name is used when the boundary is included. What name has been used previously? (Region)	An angular region is a set of points which includes the interior and the angle (boundary).
F-35 Comparison. Congruency. Larger and Smaller.	Build a basis for unit measurement.	Have models or draw representations on the board of various geometric figures, including segments, curves, simple closed curves, angles, and regions. Compare as to which is greater. Certain comparisons such as comparing segments with angles, or comparing segments with regions are obviously meaningless. When comparing segments with each other, devices such as tracings on a transparency are needed to compare accurately. If they superimpose exactly, they are congruent.	Two geometric figures are congruent if a copy of one will exactly "cover" the other. A geometric figure is larger than another figure if a part of the first figure remains uncovered when a copy of the second is superimposed on the first.

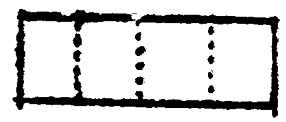
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		Make similar comparisons between segments and curves. Strings may be more useful than tracings in this comparison. Similarly compare regions of same shapes with each other and regions of different shapes with each other. In some cases, a region may have to be cut up to compare accurately. Figures that coincide exactly are said to be congruent. A figure that cannot completely be covered by a second figure is said to be <u>larger than</u> the second figure. The second figure is <u>smaller than</u> the first. An angle is said to be larger than another if its angular region is larger than the angular region of the other angle. (Reserve <u>greater than</u> and <u>less than</u> for numbers.) There is no common term for figures, neither of which is larger and yet are not congruent since they do not match without distorting or restructuring the shape.	
F-36 Common geometric figures. Square. Rectangle Circle.	Define common geometric figures.	Have students build out of wire or draw representations of quadrilaterals. Have students determine which figures can be grouped together and to define the common characteristics. A name (rhombus) may be assigned, if desirable, to all those with four congruent sides. If necessary, direct student's attention to angle characteristics. Are angles included in the quadrilaterals? No! But if the segments are extended to form rays, angles are determined. Compare the angles determined. Figures which determine four congruent angles are assigned a name — rectangle (which means four angles). Discuss the other characteristics of the rectangle. Experimentally, compare the segments to determine which are congruent.	A rectangle is a quadrilateral in which the four angles determined are congruent. A rectangle consists of two pairs of congruent segments. A square is a rectangle in which all four segments are congruent. A circle is a simple closed curve in which congruent segments are determined by the center and each point on the curve.

	POSE OR JECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		Have students cite those rectangles which may be special in the sense of use or properties. When a square is cited, have students bring out the distinguishing properties (four congruent segments) and assign a name (square). Have students point to or illustrate the common geometric figures (possibly in use in the classroom) and determine which are sufficiently important to be assigned special names. The circle should be an outgrowth. Assign its name and have students determine a unique definition. Introduce the term center and the point it names. Each point on the circle, together with the center, when used as endpoints, determine segments which are congruent to each other. Verify experimentally.	
F-37 Common geometric terms. Radius. Diameter. Side. Diagonal. Vertex. Chord.	Introduce and define commonly used terms.	Draw a representation of a circle or use a clock face as an illustration. What segments related to a circle may be sufficiently useful to need special names? Introduce illustrations of those not originated by students and introduce appropriate terms. Have students attempt to offer good definitions that limit the term to the use intended. Repeat with polygon.	A radius is the name for any segment whose endpoints are the center and a point on the circle. A chord is the name for any segment whose endpoints are points on a circle. A diameter is a chord which intersects the center of the circle. A vertex is the intersection of any two of the segments, the union of which forms a polygon.

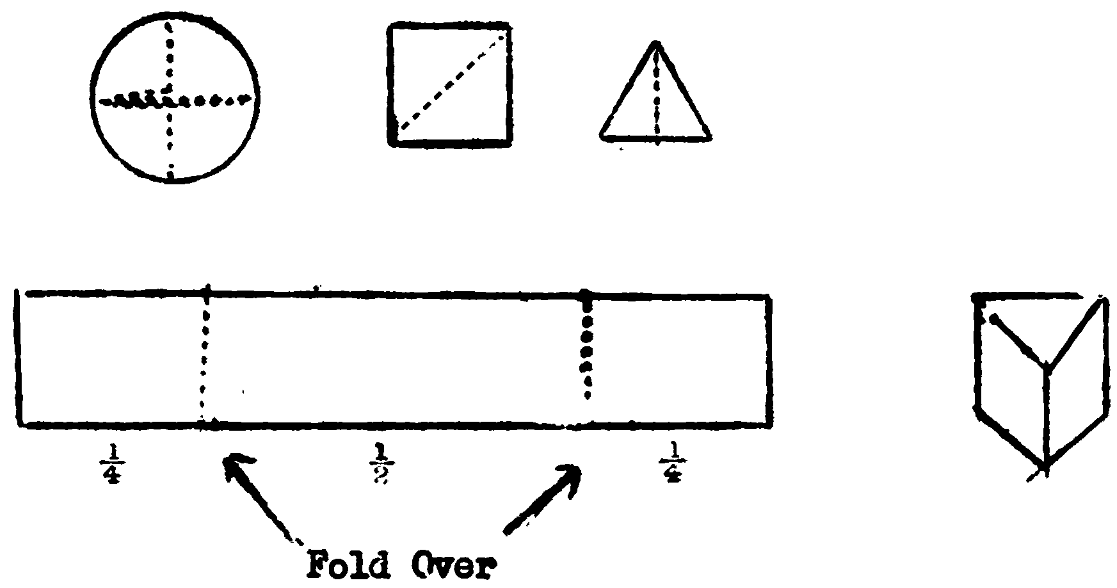
TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
			A side is any of the segments, the union of which rm a polygon.
			A diagonal is any segment whose endpoints are the vertices of a polygon, provided that segment is not a side.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
G-1 Rational Number. Unit.	Develop the concept of a unit.	There are many ways to conceptualize the word unit. What units of measurement have you used? Did you ever get weighed? What was the unit used. What else have you measured where you used a unit: 1. How about an egg? Is that a unit? Could one dozen eggs be a unit? A student? A class? A school? 2. If we say "We have been in school 15 days." One day (not one week) is our unit for counting how long we have been in school. 3. Develop characteristics of a useful unit. (Refer to D-15 Through D-22) 4. Units may be combined to form a new unit. e.g. an egg: 12 eggs in 1 dozen one inch: 36 inches in 1 yard. An egg may be considered as a unit. 12 eggs make up a new unit called a dozen.	1. The unit means ONE. 2. Units are countable.. 3. A unit is a matter of choice. 4. Units may be combined to form a new unit.
G-2 Rational Numbers Number Line	Show an abstract representation of a unit, in this case using the number line.	Have students draw number lines on the black board. Ask each one to locate zero, and then to indicate the location of one. "Is the distance from zero to one the same on each line?" No. "Must these distances all be alike?" No, any of these lengths may be used to represent this unit. If students should locate the point labeled one to the left of zero, discussion should be encouraged to show that the representation is correct but is not the typical usage.	On the number line we may select any length to represent the unit. This is indicated as the distance from zero to one. As soon as we do this the segment from one to two, from two to three and so on must be congruent to our unit segment.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING AND	SPECIFIC UNDERSTANDING FOR STUDENT
		Have di. children locate two on each number line. Ask someone else in the class, "How have they arrived at the location for two?" Discussion should lead to the generalization that successive segments must be congruent to the unit segment.	
G-3 Rational Numbers. Fractional parts of the unit. Number Line.	Show that the unit may be divided into any number of congruent parts. Define the terms numerator and denominator. Define unit fractions.	 Ask if any child can divide the unit segment into two congruent parts. Ask him to label the point he selects as point A. Ask someone to write a numeral to represent the measure of the length from zero to A.  Follow the same procedure for thirds, fourths, etc. N.B. Be sure that the children clearly understand that if our unit segment is split into four congruent parts the numeral which names one of those parts is $\frac{1}{4}$.	The unit segment may be divided into congruent parts. The numeral which names <u>one</u> of these parts has <u>one as its numerator</u> and the number of congruent segments as its denominator. This numeral is called a unit fraction.
G-4 Rational Numbers. Fractional parts of the unit. Regions.	Relate the division of the unit on the number line to the division of a rectangular region into congruent parts.	 Ask the children to draw on paper a number line and draw on the number line a rectangle whose length is the distance from zero to one.	Rectangular regions may be divided into congruent parts which are represented as fractions.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	
		Ask the children to draw on paper a number line and draw on the number line a rectangle whose length is the distance from zero to one. Cut off this rectangular region and fold it into four congruent parts. Name each part $\frac{1}{4}$. (Have the children fold the paper as in the diagram so it may be related back to the number line.  Use this rectangular region as a pattern to construct a second figure. Now cut the original rectangular region into its four parts. "Can you determine if these four parts are congruent to each other?" "Can you use these four parts to determine if the original rectangular region is congruent to the second?" Follow the same procedure for halves, thirds, etc. Represent two-thirds, no thirds, three-fourths, etc.	
G-5 Rational Number, Other Regions	Extend children's concept of unit fractions to other regions.	Have the children cut out different shapes such as circles, triangles, squares, etc. Consider each as a unit. Let the children separate the shapes into congruent parts by folding and cutting. The fraction which names any one part will have one as the numerator and the denominator will show the number of congruent parts into which the region was separated.	Any region may be separated into congruent parts and the unit fraction names one of these parts.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT



Two alternate approaches follow: (1) G-6 through G-11, and (2) G-8, G-9, G-12 through G-18.

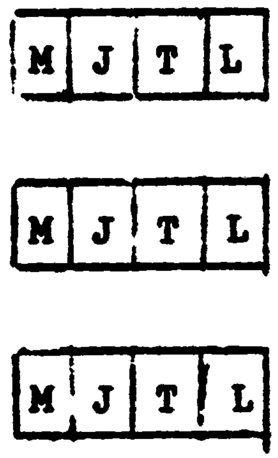
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-6 Rational Number. Addition or like unit fractions.	Review addition. Name multiples of unit fractions. Write number sentences involving the addition of unit fractions. Develop ability to add fractions.	Have children make three congruent regions and cut two into fourths. 1. Review the counting of unit fractions using the parts to represent one-fourth. Write the number sentences to describe each addition: $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$ $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$ $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{5}{4}$	1. The word "fraction" is frequently used to denote the number represented by the symbol 2. Like fractions are fractions whose denominators name the same numbers. 3. Unit fractions (the numbers can be added. Multiples of the same unit fractions (the number) can be added. When as many like unit fractions are added as the denominator indicated, the sum is one.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENTS
		2. Again using the parts representing one-fourth show: $\frac{2}{4} + \frac{2}{4} = \frac{4}{4}$ $\frac{3}{4} + \frac{3}{4} = \frac{6}{4}$ $\frac{2}{4} + \frac{3}{4} = \frac{5}{4}$ etc. 3. Have the children superimpose four one-fourths on the uncut region to show: $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4} = 1$ Separate other regions in the same way to show: $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{6}{6} = 1$ This should lead the children to generalize that: When you add as many like unit fractions as the denominator indicates, their sum will be one. 4. Have the children try to superimpose five one-fourths on the uncut region. From this they should see that they have one whole plus one-fourth. The corresponding mathematical sentence is written:	When more like unit fractions are added than the denominator indicates, the sum is greater than one. Addition of unit fractions can be written as number sentences. The addition of $\frac{3}{5}$ and $\frac{4}{5}$ may be expressed as: $\frac{3}{5} + \frac{4}{5} = \frac{(3+4)}{5} = \frac{7}{5}$ or, in general: $\frac{a}{c} + \frac{b}{c} = \frac{(a+b)}{c}$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENTS
		$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\quad \frac{1}{4} = \frac{5}{4}$ $\frac{5}{4} = \frac{4}{4} + \frac{1}{4} = 1 + \frac{1}{4} = 1\frac{1}{4}$ Extend to the addition of other unit fractional numbers whose sum is greater than one. $1\frac{1}{4}$ is usually called a mixed numeral. (This can be developed with the number line also. Refer to D-8)	
G-7 Rational Numbers. Subtraction of Like fractions.	Develop subtraction.	Regions on the number line may be used as an approach to subtraction of numbers represented as like fractions. Have the children construct a number line divided into eighths and then ask them if they can complete the number sentence: $\frac{12}{8} - \frac{7}{8} = \square$ This can be written as: $\frac{12 - 7}{8} = \square$ Extend to the case where: $\frac{3}{8} - \frac{3}{8} = \frac{3 - 3}{8} = \frac{0}{8} = 0$ Regions can be conveniently used to illustrate $\frac{5}{8} - \frac{3}{8}$ or $1 - \frac{1}{4}$ when the "take away" illustration of subtraction is used. Review the inverse concept of subtraction. Another approach to determining $\frac{5}{8} - \frac{3}{8}$ is finding $\square$ such that $\frac{3}{8} + \square = \frac{5}{8}$.	Numbers named as like fractions can be subtracted provided that $b \leq a$ in the equation: $\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$ The symbol $\leq$ means "less than or equal to".

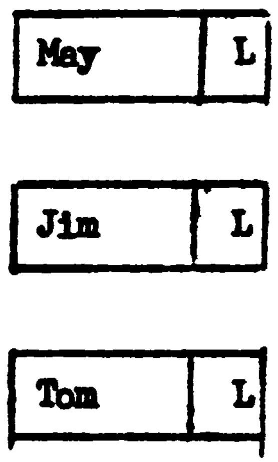
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		The difference can be found by substituting numbers in $\square$ until one is found which makes the sentence true.	
G-8 Rational Numbers. Extending the meaning of fractions.	Extend the meaning of fractions.	See G-3 1. Consider sentences such as: $3 \times \square = 12$ $4 \times \square = 7$ $5 \times \square = 2$ The numbers which make these sentences true are respectively, by the definition of fraction: $\frac{12}{3}$ $\frac{7}{4}$ $\frac{2}{5}$ 2. Other names for these numbers can be found by finding numbers which make the original sentences true. An easy case is the number 4 in the first example. When substituted the sentence becomes $3 \times 4 = 12$ which is true. 4 is, therefore, another name for $\frac{12}{3}$. Another check is dividing the numerator into the denominator to determine a quotient.	A fraction is an ordered pair of numbers (a,b) or $\frac{a}{b}$ which may be interpreted as: 1. A representation of parts of a unit in which the a, called the numerator, indicates how many parts are being considered; the b called the denominator, indicates the number of parts into which the unit has been separated. 2. A quotient of a divided by b, where a is a whole number and b is a counting number. This quotient is the missing factor, which makes the following sentence true: $b \times \square = a$.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		3. May, Jim, Tom and Lou were to share three candy bars equally. How much candy will each one get? Have each child take 3 congruent pieces of paper to represent the three bars of candy. Then have the children solve the problem by cutting the three pieces of paper. The illustrations show three possible solutions.	3. A representation of a quantity where $\underline{a}$ indicates the number of units, and $\underline{b}$ the number of parts into which that total quantity has been separated. This is expressed as: $\frac{1}{b} \times a$, which can be shown to be equivalent to $\frac{a}{b} \times 1$ or $\frac{a}{b}$.



Each get three one-fourths.

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \times \frac{1}{4} = \frac{3}{4}$$

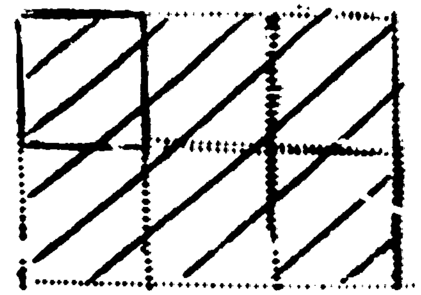


May, Jim, Tom each get three-fourths. Lou gets three one-fourths. Each of three gets $\frac{3}{4}$. One gets $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$.

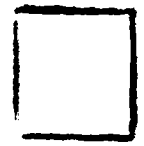
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
		[May | Jim] Each gets two-fourths plus one-fourth. [Tom | Lou] $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ [M | J | T | L] If any of these solutions are omitted by the children, direct them towards it. The children should conclude that by using any one of the three solutions each child gets the equivalent of three fourths of one candy bar. In each case the mathematical sentence is a mathematical representation of the numbers associated with objects that are joined, not added.	
G-9 Multiplication. Physical interpretation.	Give meaning to multiplication in terms of physical models. Build a basis for a rule for multiplying fractions.	1. Have students draw arrays representing products of whole numbers, bring in arrays of rectangular regions as an alternate, based on a unit region: 2 × 3 array: or × × × × × ×	Multiplication can be interpreted as a representation of separating regions or parts of a number line.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

2 × 3 array



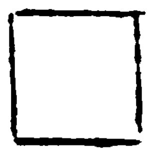
Unit Region:



Ask students for similar suggestions or implications to represent: $\frac{1}{2} \times 6$, $7 \times \frac{1}{3}$,

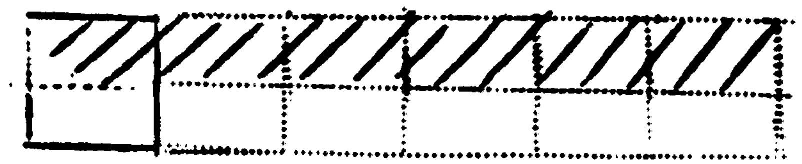
$\frac{1}{2} \times \frac{1}{3}$

Using



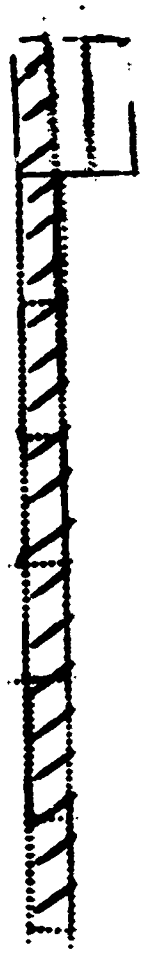
as a unit region

$\frac{1}{2} \times 6$ array:

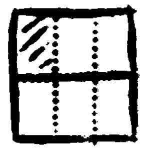


TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

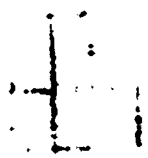
$7 \times \frac{1}{3}$ array



$\frac{1}{2} \times \frac{1}{3}$ array (shaded



[illegible] array (shaded)

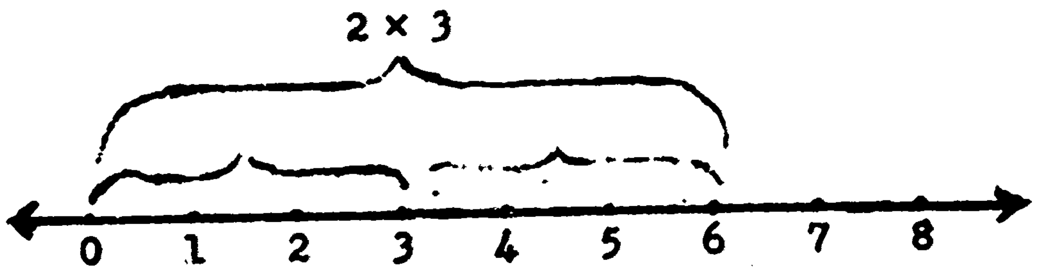


TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

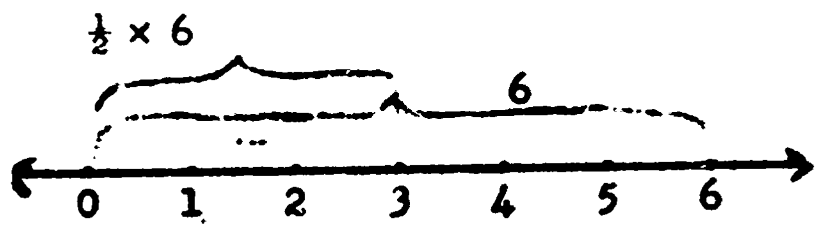
2. Expand to examples using fractions that are not unit fractions:

   $\frac{2}{3} \times 3$, $\frac{3}{4} \times \frac{1}{3}$, etc.

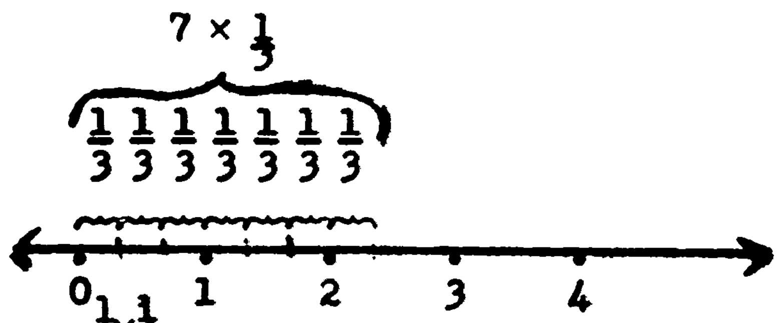
3. Have students represent physical interpretations of product of "whole" numbers on the number line.

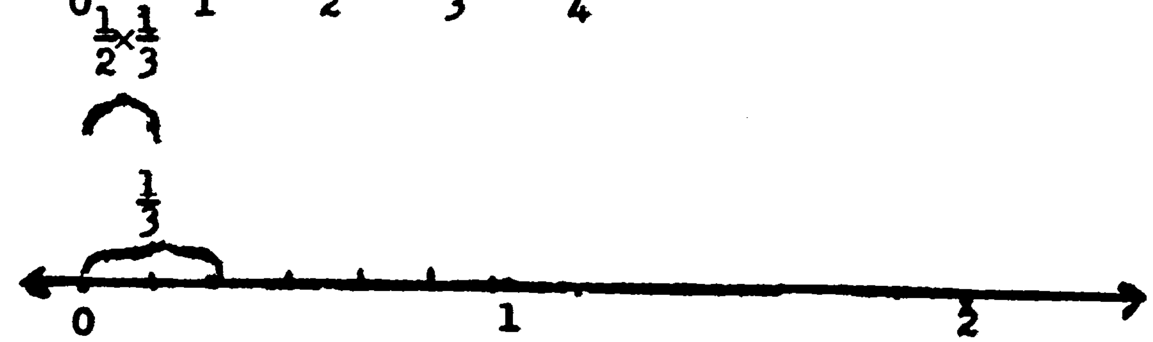


Have students seek similar representation for products involving fractions such as $\frac{1}{2} \times 6$, $7 \times \frac{1}{3}$, $\frac{1}{2} \times \frac{1}{3}$



TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT





4. Expand to examples using fractions that are not unit fractions: $\frac{2}{3} \times 3$, $\frac{3}{4} \times \frac{1}{3}$, etc.

5. Draw conclusions as to the product, retaining the form of improper fraction.

$\frac{1}{2} \times 6 = \frac{6}{2}$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$7 \times \frac{1}{3} = \frac{7}{3}$

$\frac{2}{3} \times 3 = \frac{6}{3}$

$\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVES	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-10 Multiplication. Algorithm.	Introduce an algorithm for multiplying fractions.	From the conclusions as to products obtained in G-9 have students draw generalizations, inductively, that can be used to multiply numbers represented by fractions without using physical models. Develop a precise rule or generalization including phrases such as "multiply the numbers represented by the numerator". Note the cumbersome language. Simplify the language using the understanding that any mathematical operation pertains to the numbers represented by the numerals without so stating, recognizing that an operation with numerals is meaningless.	To multiply fractions, multiply the numerators to obtain the numerator of the product, multiply the denominators to obtain the denominator of the product. Many statements such as the rule above sacrifice precision of language for shortness. To multiply a whole number and a fraction, multiply the whole number and the numerator to obtain the numerator of the product and use the denominator of the fraction factor as the denominator of the product.
G-11 Rational Numbers. Properties.	Ascertain which properties of whole numbers are valid for fractional numbers for use in future algorithms.	INDUCTIVE: Review the properties of whole numbers. Arrive at mathematical sentences which state the generalizations such as: $\triangle + \square = \square + \triangle$ $(\triangle + \square) + \nabla = \triangle + (\square + \nabla)$ $\triangle + 0 = \triangle$ Do the commutative and associative properties hold for subtraction?	The commutative, associative, and identity properties of addition and multiplication apply to fractional numbers as well as whole numbers. The distributive property of multiplication over addition or subtraction is true for "fractions".

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

Question whether these statements are true when numbers named as fractions (called fractional numbers) are substituted? when some fractional numbers and some whole numbers are substituted in the same sentence? Are the statements true regardless of the numbers substituted? Substitute many sets of numbers to check. What fraction names are other names for 0 and can be used instead of 0 to facilitate addition?

DEDUCTIVE:

Use specific cases to start for each of the properties and generalize if students are able:

$$\frac{3}{4} + \frac{2}{4} = \frac{2}{4} + \frac{3}{4} \qquad \frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}$$

$$\frac{(3 + 2)}{4} = \frac{(2 + 3)}{4} \qquad \frac{a + c}{b} = \frac{c + a}{b}$$

(Since the numerators represent whole numbers and the commutative property is true for addition of whole numbers, the sentence will be true regardless of the numbers chosen)

Similarly:

$$\left[\frac{1}{5} + \frac{4}{5}\right] + \frac{3}{5} = \frac{1}{5} + \left[\frac{4}{5} + \frac{2}{5}\right]$$

$$\frac{1 + 4}{5} + \frac{3}{5} = \frac{1}{5} + \frac{4 + 3}{5}$$

$$\frac{(1 + 4) + 3}{5} = \frac{1 + (4 + 3)}{5}$$

2

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		and: $\frac{3}{8} + 0 = \frac{3}{8}$ $\frac{3}{8} + \frac{0}{8} = \frac{3}{8}$ $\frac{3+0}{8} = \frac{3}{8}$	

G-12 to G-18 is an Alternate Approach for G-6, G-7, G-10 and G-11. Proceed from G-5 to G-8 and G-9 and thence to G-12.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-12 Rational Numbers. Definition. Unit Fraction.	Define fraction.	Obtain regions of various shapes. Cut each region into congruent parts. Cut different regions into different numbers of parts. Retain a copy of the original to indicate a unit. Show students a unit and a part. Determine the number of parts needed to form a unit. ~~Ex:~~ When the union of three congruent parts forms a unit, we can express their related numbers in terms of addition as: $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ Since one factor in multiplication indicates the number of times an addend is used to arrive at a final sum, the above expression can be written in multiplication form as : $3 \times \frac{1}{3} = 1$ Similarly for sixths: $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$ $6 \times \frac{1}{6} = 1$	A unit fraction is expressed as one over a whole number. A unit fraction represents a number which (1) when multiplied by a number represented by its denominator equals one or (2) when used as an addend a number of times equal to the number represented in the denominator equals one.

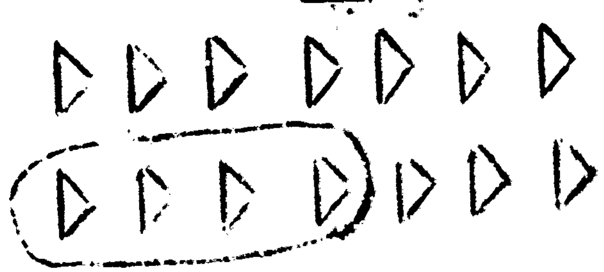
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-13 Definition. $\frac{a}{b}$	Define common fractions for ease in addition.	Define $\frac{3}{4}$ as a short way of writing $3 \times \frac{1}{4}$. Have students anticipate and illustrate corresponding definitions for $\frac{2}{3}$, $\frac{7}{8}$, $\frac{1}{5}$. Have students attempt to generalize.	$\frac{a}{b} = a \times \frac{1}{b}$ $\frac{1}{b} = 1 \times \frac{1}{b}$
G-14 Definition. Like fractions.	Prepare for addition.	Fold a region into eighths. Cut along some folds to obtain different size parts. Name the numbers associated with these parts by comparing to a copy of the original unit region. What is alike in all these names? Introduce the term "like fractions."	
G-15 Properties	Define those properties which will be valid with fractions.	Review the properties of whole numbers that have been used to determine addition and multiplication algorithms. Discuss with students whether they think these properties should be true for the new numbers named as unit fractions. Review the value of these properties in building algorithms. (For this reason the properties will be assumed to be true for unit fractions as well as whole numbers.)	The following properties of whole numbers are true for numbers named as unit fractions: 1. Closure for addition. 2. Closure for multiplication. 3. Commutative in addition. 4. Commutative in multiplication. 5. Associative in addition. 6. Associative in multiplication. 7. Identity in addition. 8. Identity in multiplication. 9. Distributive property of multiplication over addition. 10. Distributive property of multiplication over subtraction

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-16 Addition.	Introduce addition algorithms for fractions.	1. Review that only numbers are added. Fractions are numerals. Can fractions, therefore, really be added? Adding fractions is a common term which means adding the numbers represented by the fractions.	1. The word "fraction" is frequently used to denote the number represented by the symbol. "Adding fractions" means adding the numbers named.
		2. Take two like fractions at random (suggested by students) as addends. Represent each addend as a product. EX. – $\frac{3}{8} + \frac{1}{8} = (3 \times \frac{1}{8}) + (1 \times \frac{1}{8})$ Solicit suggestions as to useful properties for addition. Apply the distributive property: $\frac{3}{8} + \frac{1}{8} = (3 + 1) \times \frac{1}{8}$ If necessary, commute first to arrange numbers in accepted order: $\frac{3}{8} + \frac{1}{8} = (3 \times \frac{1}{8}) + (1 \times \frac{1}{8})$ $= (\frac{1}{8} \times 3) + (\frac{1}{8} \times 1)$ $= \frac{1}{8} \times (3 + 1)$ Addition of whole numbers is suggested. Obtain either $4 \times \frac{1}{8}$ or $\frac{1}{8} \times 4$. Write the fraction name for this number $\boxed{\frac{4}{8}}$ which is the sum sought.	2. Like fractions are added by adding the numbers represented by the numerators and expressing the sum over the same denominator as the addends.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-17 Subtraction.	Introduce subtraction algorithms for fraction.	Take two like fractions and express as a difference. (For first examples select the first fraction greater than the second). Express each fraction as a product. EX: $\frac{3}{4} - \frac{2}{4} = (3 \times \frac{1}{4}) - (2 \times \frac{1}{4})$ Solicit useful approaches from the children. Use the distributive property. $\frac{3}{4} - \frac{2}{4} = (3 - 2) \times \frac{1}{4}$ $= 1 \times \frac{1}{4}$ $= \frac{1}{4}$ If numbers are selected by random (such as students offering suggestions) it will be noted through attempted subtraction that not all pairs of fractions have a difference in the set of numbers known to the student at the time.	Like fractions are subtracted by subtracting the numbers represented by the numerators and expressing the difference over the same denominator as the fractions subtracted. The set of like fractions is not closed for subtraction.
G-18 Addition. Sums ≥ 1.	Introduce addition leading to sums of one or mixed numbers.	1. Ask students how many one-fourths must be added to obtain one. Check by actually adding $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = (1 \times \frac{1}{4}) + (1 \times \frac{1}{4}) + (1 \times \frac{1}{4}) + (1 \times \frac{1}{4})$ $= (1 + 1 + 1 + 1) \times \frac{1}{4}$ $= 4 \times \frac{1}{4}$ $= \frac{4}{4}$	1. The distributive property reconfirms that when as ~~many~~ unit fractions are added as the denominator indicates, the sum is one.

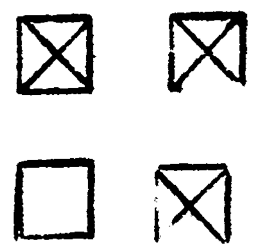
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		2. Display a unit region or a unit on the number line. Pre-cut copies of the unit into congruent parts. Display a number of these congruent parts of the unit which is greater than the denominator. Have students join the pieces. Perform the corresponding addition for the numbers represented by these concrete materials.	2. When more like unit fractions are added than the denominator indicates the sum is greater than one.

EX. - Unit 



$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4})$



$$\underbrace{(1 + 1 + 1 + 1) \times \tfrac{1}{4}}_{(4 \times \frac{1}{4})} + \underbrace{(1 + 1 + 1) \times \tfrac{1}{4}}_{(3 \times \frac{1}{4})}$$

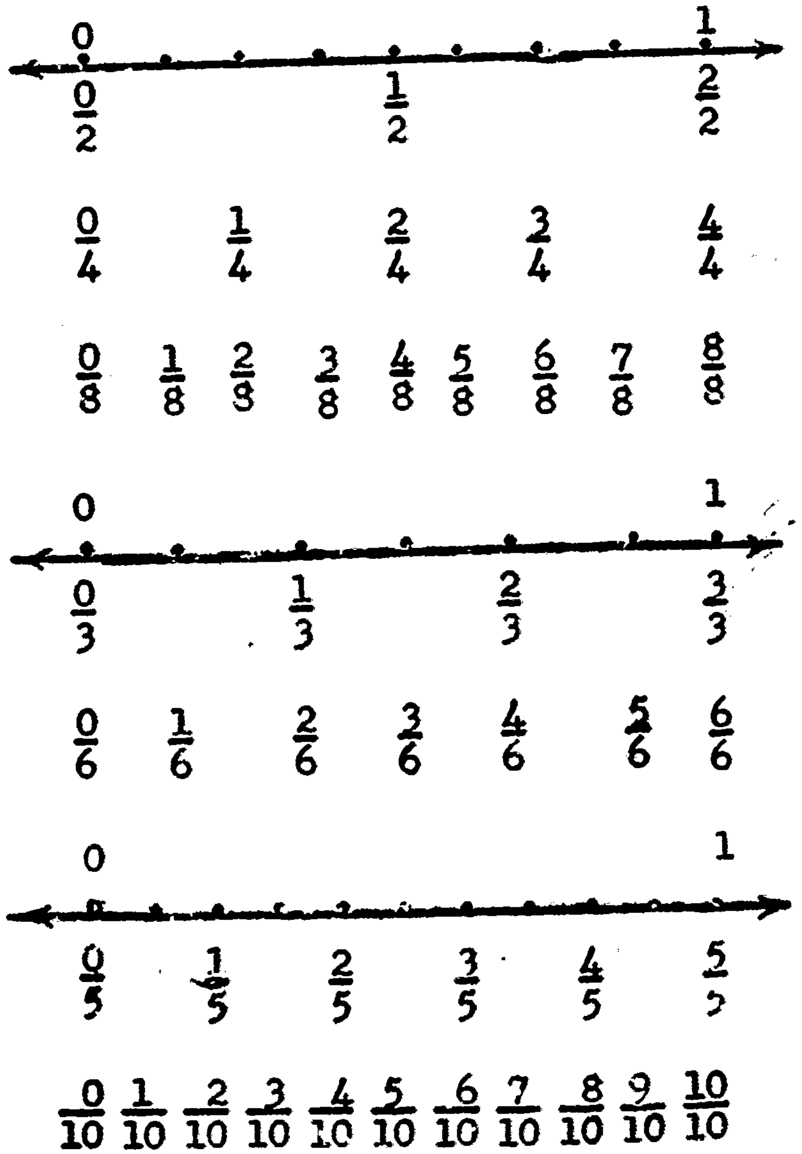
$$1 + \frac{3}{4}$$

$$1\frac{3}{4}$$

If students add like fractions by methods other than the distributive property, simply accept.

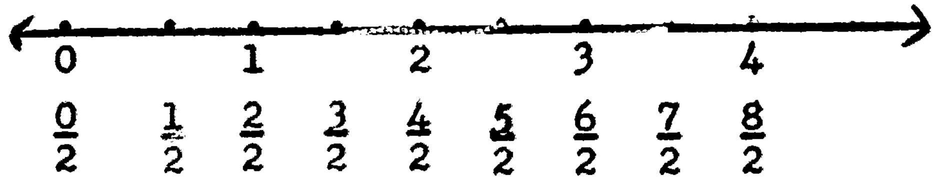
End of Alternate Sequence

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
G-19 Rational Numbers. Equivalent Fractions .	Understand equivalent fractions. Relate the set of whole numbers to the set of rational names.	1. Have each child prepare three number lines, one below the other, the unit zero to one being the same length, and mark their number lines as shown in the diagram. (The zero points must be in alignment.)	1. There are many names for the same point on the number line. It is convenient to be able to express these points by different names.



Ask the children to make a list for the other names they find for the point at one. (2/2, 3/3, 4/4 etc.) In the same way make a list of fractions which identify the point 1/2, (1/2, 2/4, 3/6 etc.) Follow the same procedure with other points.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		2. Review the fact that one times any number is that number. Note to students that this generalization will also hold for rational numbers. e.g. $1 \times \frac{1}{2} = \frac{1}{2}$. Observe from the number line that one can be expressed in many ways, eg. $\frac{2}{2}$, $\frac{3}{3}$, $\frac{5}{5}$ etc. Have the children experiment in the multiplication of 1 and a fractional number substituting names for one. e.g. $\frac{2}{2} \times \frac{1}{2} = \frac{2}{4}$, $\frac{3}{3} \times \frac{1}{2} = \square$, $\frac{5}{5} \times \frac{1}{2} = \square$, etc. Observe, from the number line that these products all name the same point, therefore, they are equivalent fractions. Calculate to find other sets of equivalent fractions.	2. The identity element for multiplication can be used to find sets of equivalent fractions. Equivalent fractions are different names for the same rational number. $\frac{a}{a} = 1$, where a is any whole number not equal to zero.
		3. Have each child make a number line, from zero to four, and then name the points from $\frac{0}{2}$ to $\frac{8}{2}$.	3. The set of whole numbers is contained in the set of rational numbers.



Observe that $\frac{2}{2}$ is another name for 1, $\frac{4}{2}$ is another name for 2, $\frac{6}{2}$, for 3. We may show this mathematically by factoring:

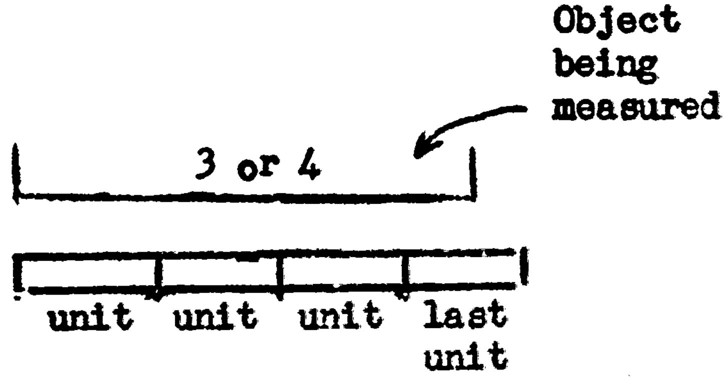
$$\frac{10}{2} = \frac{2 \times 5}{2 \times 1} = \frac{2}{2} \times \frac{5}{1} = 1 \times \frac{5}{1} = 5$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		We can also show mathematically that: $5 = 1 \times \frac{5}{1} = \frac{2}{2} \times \frac{5}{1} = \frac{2 \times 5}{2 \times 1} = \frac{10}{2}$ From other similar calculations the children will observe that any whole number may be expressed as a rational number. Therefore, the set of whole numbers is contained in the set of rational numbers.	
G-20 Rational Numbers. Ordering fractions.	1. Order unit fractions.	1. Supply each child with strips of different colored paper all the same length. Cut one color into halves, the second into thirds, etc. Label each piece with the appropriate unit fraction and then order according to size. Use the generalization developed to order sets of unit fractions. e.g. $\frac{1}{15}, \frac{1}{84}, \frac{1}{35}, \frac{1}{57}, \frac{1}{21}, \frac{1}{89}, \frac{1}{11}$, etc. These can be ordered both ways using $<$ or $>$.	1. As the denominator of a unit fraction increases the part of the unit represented by the fraction decreases in size.
	2. Order fractions with like denominators.	2. Have each child use a number line divided into sixteenths and label the parts. Observe that the larger the numerator the larger the part the fraction represents.	2. Fractions of like denominators increase in value as the numerator increases in value.
	3. Order fractions with unlike denominators.	3. Ask the children to order the fractions $\frac{2}{3}$ and $\frac{3}{4}$ using the symbol $<$. Some children will probably use the number line, some may use the colored paper strips, others may use the identity element to rename these into fractions with like denominators. e.g. $\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$, $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$. Then $\frac{2}{3} < \frac{3}{4}$. Apply this renaming approach to such fractions as $\frac{5}{9}$ and $\frac{7}{10}$.	3. Fractions with unlike denominators may be ordered more easily by renaming them as fractions with like denominators.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
G-21 Rational Numbers. Ordering fractions.	1. Order unit fractions.	1. Supply each child with strips of different colored paper all the same length. Cut one color into halves, the second into thirds, etc. Label each piece with the appropriate unit fraction and then order according to size. Use the generalization developed to order sets of unit fractions. e. g. $\frac{1}{15}, \frac{1}{84}, \frac{1}{35}, \frac{1}{57}, \frac{1}{21}, \frac{1}{89}, \frac{1}{11}$, etc. These can be ordered both ways using $<$ or $>$	1. As the denominator of a unit fraction increases the part of the unit represented by the fraction decreases in size.
	2. Order fractions with like denominators.	2. Have each child use a number line divided into sixteenths and label the parts. Observe that the larger the numerator the larger the part the fraction represents.	2. Fractions of like denominators increase in value as the numerator increases in value.
	3. Order fractions with unlike denominators.	3. Ask the children to order the fractions $\frac{2}{3}$ and $\frac{3}{4}$ using the symbol $<$. Some children will probably use the number line, some may use the colored paper strips, others may use the identity element to rename these into fractions with like denominators. e.g. $\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$, $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$ Then $\frac{2}{3} < \frac{3}{4}$. Apply this renaming approach to such fractions as $\frac{5}{9}$ and $\frac{7}{10}$.	3. Fractions with unlike denominators may be ordered more easily by renaming them as fractions with like denominators.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
		A technique usable with more difficult problems is illustrated below. $\frac{13}{57} = \frac{13\times19}{57\times19} = \frac{247}{57\times19}$ $\frac{4}{19} = \frac{4\times57}{19\times57} = \frac{228}{19\times57}$ Note that 19×57 and 57×19 both name the same number; their product need not be calculated. Since $228<247$, then $\frac{4}{19} < \frac{13}{57}$.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
MEASUREMENT H-1 Need	Motivate need for measurement.	Use an example where 2 things are compared but cannot be done directly: Johnny wants to check whether the pitcher's mound is the same distance from home plate on his Little League field as on the school field. Can he use a short piece of string, which is the only thing available, to compare the two? Lead to the idea that measurement is counting the number of congruent units needed to form that which is being measured.	Measurement is needed to compare when direct comparison is impossible or not convenient. Measurement is counting the number of congruent units needed to "cover" the figure to be measured.
H-2 Length	Clarify the nature of a unit needed for measuring curves.	Using an example such as a coathanger, what kind of unit must be used to determine the length of the wire? Must the unit be straight? Must it be a piece of wire? Must it be curved? Must the hanger be straightened before being measured? Could the unit be a stick? Could a ball be used? A rubber band? A quart of water?	Any curve of finite length can be used as a unit to measure a longer curve or segment. A segment can be used as a unit to measure a longer curve or segment. To measure a one-dimensional figure, a one-dimensional unit must be used.
H-3 Perimeter	Define perimeter.	Use models of simple closed curves such as wire or drawings on the board. Discuss the length of the curve and possible ways of determining the length. Indicate that this length is called perimeter.	The measure of the length of a simple closed curve is called its perimeter.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-4 Approximate Nature	Introduce the approximate nature of measurement.	Ask student what he would do if there were a small piece of wire to be measured that is too small for the unit to measure. Are some measures more exact than others? Is any measure exact? 	In measurement, the last unit is counted if at least half of the unit is used in "covering" the object being measured.
H-5 Common (Standard) Units	Review standard units and build a feeling for the actual size.	Ask students to suggest names of units of length with which they are familiar. Have students draw segments to represent these units. Ask students to represent the mile. Let him draw the conclusion that it cannot be represented on paper or the board by a segment that is actually one mile long. Have students actually examine models of the units.	Some commonly accepted standard units of length are inch, foot, yard, meter, centimeter, and mile.
H-6 Notation	Introduce precise terminology and symbolism.	Draw a line segment AB or curve AB. Have a student select a unit (which may be an arbitrary segment) and measure the segment or curve. Indicate to the student that the number of units used is called "Measure." Results of the measuring might be:	A measure is a whole number which indicates the number of times a unit was counted. A measurement includes the measure and the unit and indicates the "size." The symbol "≈" means "is approximately."

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
		The measure of $\overline{AB}$ is 6 and symbolized as $m\overline{AB} = 6$. The measurement or length of $\overline{AB}$ is approximately 6 units and symbolized as length of $\overline{AB} \approx 6$ units.	
H-7 Standard Unit Relationship	Introduce alternate ways of expressing common equivalent measures.	Discuss with the students the relationships that they already know. Supply those that they do not know. Indicate that these are equivalent expressions. (Ex. 9 ft. may be written as 3 yds.) 80 in. may be expressed as 6 ft. 8 in.) Discuss the advantages of various equivalent expressions. The choice depends on the requirements of the situation.	The standard units of length are related as follows: 12 in. = 1 ft. 3 ft. = 1 yd. $16\frac{1}{2}$ ft. = 1 rod 5280 ft. = 1 mile 100 cm. = 1 m.
H-8 Precision	Recognize the limitation of measurement.	Take strips of paper or sticks. Arrange in lengths from 6'-7/8" to 7'-1/8". Have the students measure with a ruler (which consists of a series of congruent one-inch units laid end to end.) Students will probably say that they are 7" long. Compare the objects to show that they are not all the same length. Discuss the possible actual lengths for which the student will report a measurement of 7". (The variation of size of pieces for the experiment may differ according to the measuring skill of the students.)	The error between the actual length and the measurement may be as much as $\frac{1}{2}$ unit in either direction. (too large or too small).

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-9 Cumulative error.	Recognize the limitation of measurement.	Suppose a machine consistently cuts strips of paper slightly longer than the measurement prescribed. What happens when they are put together? Take the 10 longest strips of paper from the previous example and lay them end to end. Predict how long they should be. Have the class actually measure them. Was the prediction correct? Now take all the strips, long and short, and place them end to end and measure. Notice how the errors tend to cancel one another.	The error in measurement may be more obvious when a series of measured objects are put together.
H-10 Rounding.	Appreciate that the direction of rounding depends upon the application.	You need a board 7 ft. 2 in. long to build a book shelf. Boards are sold only by the foot. What length of board would you buy? You need boots, which are sold only in the whole sizes. Size six is a little too small. Size seven is too large. Which would you buy? If a window frame is exactly 36 inches long, could you use a pane of glass that was actually $\frac{1}{8}$" too long when it was measured and cut? If it had been cut $\frac{1}{8}$" too short, could you fit it into the frame? Suppose you were cutting glass 39" long and you did not know its use. Would it be better to cut it a little larger or smaller since you could not cut it exactly? Why? (Have students justify their answer.) How do we usually round numbers in convenient measurement? (Read it to the nearest unit.)	Measurements are usually rounded to the nearest unit. Some applications require that all measurements be rounded "up" or all rounded "down."

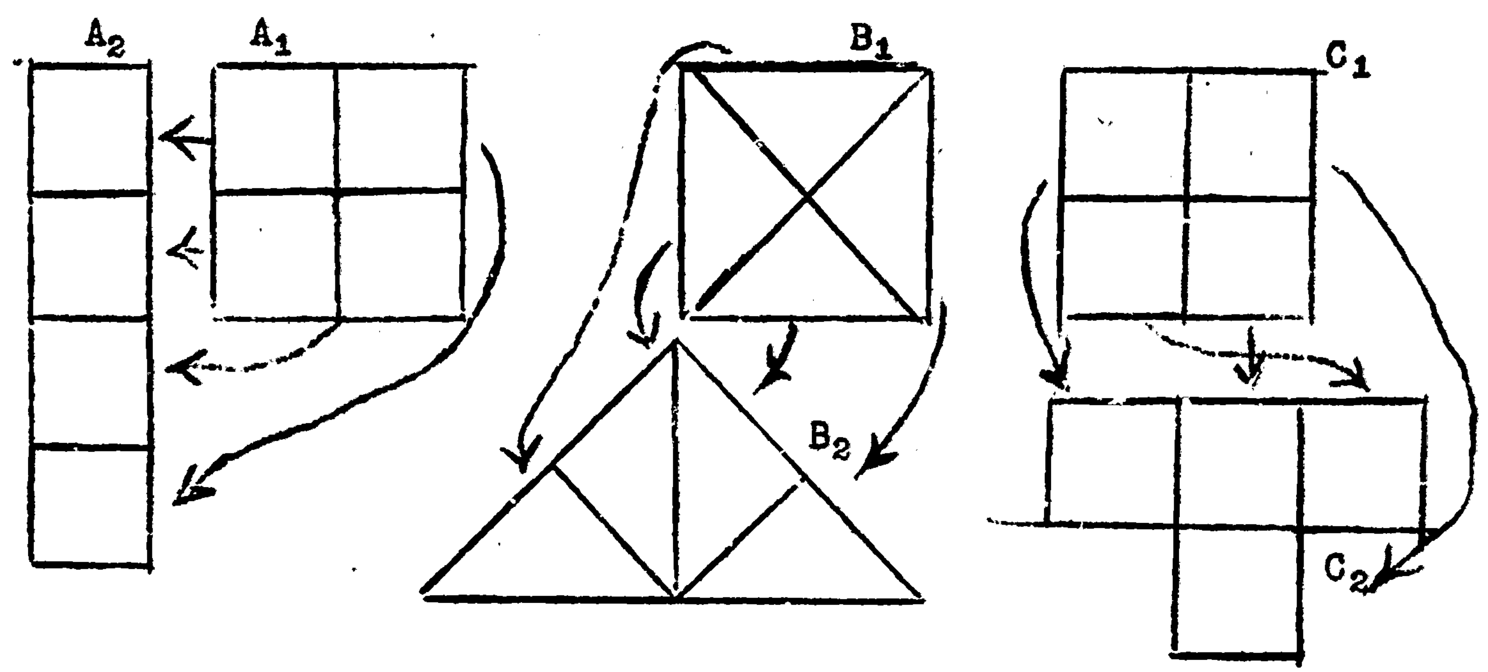
TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-11. Standard Unit Preference.	Lead students to the best choice of a unit for a particular situation.	Discuss the meaning and accuracy of statements such as: 1. The sun is 5,892,480,000,000 inches away. (The number is too large for the size to be appreciated. The unit implies measurement to the closest inch). Is miles a better choice of unit? Is there a still better choice? 2. The desk top is 2 feet long. (The unit implies the desks tops were cut to somewhere around 2 feet long and may be in error by several inches. They are probably measured more accurately and the unit should indicate the precision.) 3. 36 inches of yarn is the same as 1 yard of yarn. 8.00 ft. = 8 ft. Measurements expressed in different units imply different degrees of precision. Implications are different. For example 36 inches implies precision to the nearest inch ($35\frac{1}{2}$ inches to $36\frac{1}{2}$ inches) whereas 1 yard implies precision to the nearest yard ($\frac{1}{2}$ yard to $1\frac{1}{2}$ yards).(8.00 ft. implies a unit of one hundredth foot. 8 ft. implies a unit of one foot.) Desirable precision is usually determined by the use. Measurements for a watch are much more precise than for cloth in making dresses.	Large units are used to limit measure to small numbers whose size is readily appreciated. Units must be sufficiently small to convey to the reader or listener the precision of the measurement. Desirable precision is usually determined by the use.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLE	SPECIFIC UNDERSTANDING FOR STUDENT
H-12 Perimeter Formulas	Develop common formulas.	Discuss convenient ways of determining perimeters of polygons. Include finging the lengths of each side separately and adding their measures. Symbolize each measure (not the segment) by a letter. Represent the sum by a formula involving addition. Review use of the distinction between addition and union. Segments are joined (union of sets). Then measures (numbers) may be added. If the length of the frame of the bulletin board is measured by counting foot rulers and found to be 22 ft. (to the nearest foot); and the measure of the height (measured in inches) is found to be 13, ask the students to determine the perimeter. If students write $p = 22 + 13 + 22 + 13$, discuss the meaning of p. Would a sum of 70 be meaningful? Should the unit be inches or feet? Would a sum of 44 ft. 26 in. (or 46 ft., 2 in.) be meaningful? Such an expression implies the total length is between 46 ft. $1\frac{1}{2}$ in. and 46 ft. $2\frac{1}{2}$ in., which is misleading since the length was only measured to the closest foot, not to the closest inch. (46 ft. is the best acceptable answer.)	The measure of any polygon (called the perimeter) is equal to the sum of the measure of the lengths of all the segments which form the polygon. If p represents the measure of the perimeter and other letters represent the measures of the segments, then $p \text{ (of triangle)} = a + b + c$ $p \text{ (of quadrilateral)} = a+b+c+d$ For p to be meaningfully interpreted as length (in terms of units), each of the segments must be measured by using the same unit.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-13 Length. Figures whose parts are not segments.	Measure "curves."	Draw some representations of "curves" (no segments) on the chalkboard. Ask students to determine the length. Select a unit (such as foot or inch). Will a segment do as a unit? Students will observe the need to cut string the length of the unit and bend these pieces to fit around the curve. Have students measure a jumping rope. Some are likely to "straighten" the rope to form a segment rather than distort the unit.	Units are distorted in shape but not in "size" in determining a measure. A unit segment is a common unit in measuring "curves."
H-14 Length. Circle.	Introduce the formula for the perimeter off a circle.	Take various circular objects (dishes, wheels, etc.) Have students draw representations of diameters and measure the lengths. Find the circumference by wrapping a string around and measuring the string or by rolling the object one revolution and measuring the path. Divide the measure of diameter of each object into its circumference. Compare results. Notice the constancy and value. Introduce the name pi for this number and its symbol, π. Write the sentence obtained and its inverse relationship (definition of quotient): $\frac{c}{d} = \pi$ $\quad$ $d\pi = c$ or $d \times \pi = c$ Indicate this as a common formula for circumference and review the need for like units of mmeasure for circumference and diameter.	"Circumference" is another name for the perimeter of a circle. The quotient of the measure of the circumference of any circle divided by the measure of its diameter is a unique number, called pi (π). Pi has a value of approximately $\frac{22}{7}$. The formula for the circumference of a circle is $c = \pi d$, where c and d are the measures of its circumference and diameter.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	
H-15 Area.	Build concept of area.	Review the distinction between a segment (set of points), a measure (number), and length (size). Draw similarities in region and area.	Area is the measurement of a region. It is not a geometric figure or a set of points.
H-16 Area Units.	Determine acceptable units for area. Understand the concept of direct measurement in area.	Ask students to determine the area of the desk top. What unit could be used? Could a piece of string be used? How many would be needed? Could congruent pieces of construction paper be used? Must they be the same color? How many are needed? What do you do when a piece hangs over the side? Do you throw away left-over pieceswhen you cut them? Must the construction paper be a square region? Could they be triangular? Try congruent triangular regions. (It works!) Could paper plates be used as units? How about a quart of milk?, an hour?, a pound? Stretch an elastic band around three pins on the bulletin board to represent a triangle. What unit can be used to measure the region? Have students use construction paper units to determine the number needed to cover. Use previously prepared triangular pieces that come out "even". Use square regions. Use rectangular regions. In one example mix units of different sizes. Student reaction should reinforce the need for congruent units in any one measurement. Could any size square unit be used as a unit?	The unit of area is any region formed by a simple closed curve. Measure of a region means counting the number of units needed to cover the region. The units that are most useful are triangular and quadrilateral regions because they can be conveniently arranged without gaps between them.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-17 Area. Applications.	Relate direct measurement to life problems.	Have students find examples, or supply if necessary, where units are <u>counted</u> to determine the area. <u>Example</u>: Counting tiles on a floor, ceiling tiles, shingles for a roof or outside wall, mosaics on a table or dish.	Some applications utilize direct measurement (counting).
H-18 Standard Units. Relation to Length.	Relate units of area to units of length. Develop readiness for area formulas.	Review that the unit of area must be a closed region. Ask student what shape standard unit he would pick if given a choice? Which region would he pick? triangular?, rectangular?, square? The most commonly used figure is the square region. Refer to floor and ceiling tiles to verify that these are most common shapes. What size square region would you select as a standard unit to determine the area of a book cover?, kitchen floor?, farm?, state? What is the length of the side of the unit square region? Only the acre seems to lack a unit length to determine a square unit. Have students cut out regions from newspapers to represent as many standard units as they can. Have students cut out square regions whose side is one foot long. Cut into 4 congruent regions. Reassemble each into various shapes. Could each represent a square foot? Must a region of one square foot be square? Examples:	Most standard units for area are based on a square whose side has one standard unit of length. Standard units include square inch, square foot, square yard, square mile, square centimeter, square meter, acre. The unit of area such as square inch is not necessarily square.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
			
H-19 Region. Ordered Array of unit regions.	Develop readiness for area formulas.	Give the students a piece of construction paper and have them measure it off into one inch squares. (Ruler or one inch cardboard region may be used.) Point out that the arrangements are in ordered arays. Review the short method of determining the numbers in an ordered array.(multiplying)	Many regions can be subdivided into an ordered array of unit regions. A short way of determing the number of units is through multiplication.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
H-20 Formulas for Area.	Introduce formula for area of rectangular regions.	Ask students for suggested units for determining the area of a sheet of paper. Is there a convenient instrument which measures area? No. Could a ruler be used? No. Why not? A ruler measures length. Could such information be used in turn to determine area? Measuring the length of the long segment (or side) indicates the number of columns in an ordered array. Measuring the length of the short side indicates the number of rows. The product is determined by multiplying. After several specific examples, generalize and substitute letters to generate a formula.	The measure of a rectangular region (A) may be determined by multiplying the measure of the longer side (L) by the measure of the shorter side (W). Areas are often determined indirectly by measuring length. The formula for area of a rectangle is frequently written as $A = LW$.

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TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
RATIONAL NUMBERS. I-1. Simplification. Fractions.	Establish a basis for a standard form for answers in fractional forms.	Review prime factorization. (See E-41) Six students handed in the following answers to a problem: Tom, $\frac{4}{8}$; Mary, $\frac{5}{17}$; Betty, $\frac{7}{29}$; John, $\frac{40}{80}$; Joe, $\frac{13}{26}$; Jane, $\frac{5}{10}$. Four answers were correct. Who had correct answers? Students will probably try to relate the numbers to a simpler form for comparison by methods such as the number line. (See G-1) Encourage students to come up with methods of comparison such as finding the simplest form by looking for common factors. Suggest possible ways such as: 1. Find a common factor of the numerator and denominator and rename the numbers as products of the common factor. Repeat, if necessary, to obtain simplest form. For example: $\frac{40}{80} = \frac{4 \cdot 10}{8 \cdot 10}$ $= \frac{4}{8} \cdot \frac{10}{10}$ * $= \frac{4}{8} \cdot 1$ $= \frac{4}{8}$ $= \frac{1 \cdot 4}{2 \cdot 4}$ $= \frac{1}{2} \cdot \frac{4}{4}$ * $= \frac{1}{2} \cdot 1$ $= \frac{1}{2}$	The simplest form (or lowest terms) of a fraction is one in which the numerator and denominator have no common factor other than one. (See G-6 for definition of fraction) Changing fractions to the simplest form helps comparing results to determine whether they are equivalent. Prime factorization is a useful tool in determining the simplest form of a fraction. The raised dot is sometimes used as a symbol for multiplication. When no operation symbol is indicated between a letter, used as a place holder, and ~~another~~ letter or numeral, the operation of multiplication is implied.

(I-1 continued)

2. Find the greatest common factor and rename as in 1.

$$\frac{40}{80} = \frac{1 \cdot 40}{2 \cdot 40}$$
$$= \frac{1}{2} \cdot \frac{40}{40} \; *$$
$$= \frac{1}{2} \cdot 1$$
$$= \frac{1}{2}$$

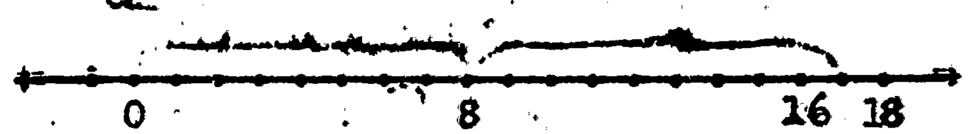
3. Write numbers as prime **factorizations.**

$$\frac{40}{80} = \frac{2 \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$$
$$\frac{40}{80} = \frac{\_}{2} \cdot \frac{2 \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5} \; * \quad \text{or} \quad \frac{40}{80} = \frac{\_}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{5}{5} \; *$$

Discuss which other number should go in the numerator. Some students will probably suggest a zero, in which case the teacher suggests multiplying them together to check with the original fraction. Students should conclude that the only possible substitution is one.

$$\frac{40}{80} = \frac{1}{2} \cdot \frac{2 \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5} \quad \text{or} \quad \frac{40}{80} = \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{5}{5}$$
$$= \frac{1}{2} \cdot 1 \qquad\qquad = \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \cdot 1$$
$$= \frac{1}{2} \qquad\qquad = \frac{1}{2}$$

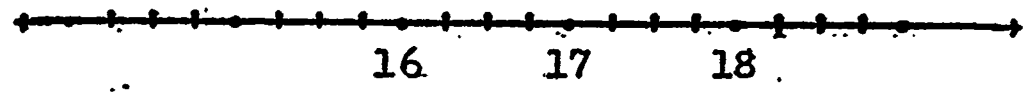
A need for this approach may be more evident when a student is asked to change numbers such as $\frac{51}{68}$ to its simplest form.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-1 continued)		*While patterns are noticeable, in each case determination of the simple form depends on the idea that since $\frac{1}{2} \cdot \frac{40}{40} = \frac{1 \cdot 40}{2 \cdot 40}$, similarly $\frac{1 \cdot 40}{2 \cdot 40} = \frac{1}{2} \cdot \frac{40}{40} = \frac{1}{2} \cdot 1 = \frac{1}{2}$ (Right and left of the first mathematical sentence is reversed in the second.) Ask students for suggestions as to a suitable "simplest form" for $\frac{16}{4}$ or $\frac{22}{8}$. ($\frac{4}{1}$ and $\frac{11}{4}$, respectively.)	
I-2. Simplification. Fractions to mixed numerals or whole numbers.	Relate fractions to whole numbers for better intuitive meaning.	Use verbal problems that involve division and which can be expressed as a fraction whose numerator is greater than the denominator such as: Mary had 18 glasses of punch to serve 8 girls. How many glasses of punch would each girl be served? How much could each girl get using all the punch? $\frac{18}{8}$ or $18 \div 8$ or $8\overline{)18}$ Using methods previously developed students would respond to the first question: " 2 glasses each with 2 glasses remaining. Proceed to the second question. A possible concrete approach is to find this by using a number line. 	1. Fractions with numerators equal to or greater than the denominator can be renamed as a whole number or a whole number and a fraction less than one where the fraction is in simplest form. 2. A fraction with a zero numerator can be simply renamed as zero. ($\frac{0}{4} = 0$) 3. A mixed number is a number whose numeral indicates the sum of a whole number and a fraction.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(I-2 continued)

What can we do with the remaining 2 cups? Can they be divided equally?



Follow up the number line with the algorithm pointing out the analogies. (See E-24)

$$8\overline{)18} \quad 2\frac{2}{8} \text{ or } 2\frac{1}{4}$$
$$\begin{array}{r|l} 16 & 2 \\ \hline 2 & \end{array}$$

Another approach is

$$\frac{18}{8} = \frac{16+2}{8}$$
$$= \frac{16}{8} + \frac{2}{8}$$
$$= 2 + \frac{2}{8}$$
$$= 2 + \frac{1}{4}$$
$$= 2\frac{1}{4}$$

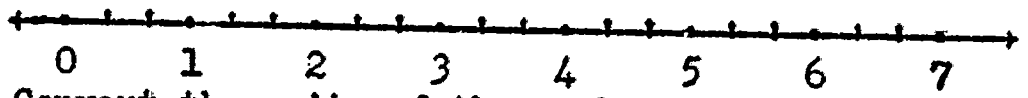
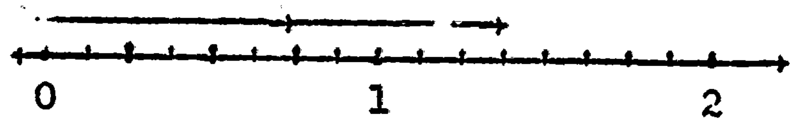
Review the number line approach in G-19 to show that $\frac{0}{4} = 0$.

Include examples such as $1\frac{8}{10}$, $2\frac{5}{4}$, $3\frac{4}{4}$, $2\frac{5}{10}$ when defining mixed numbers. (See also G-6 and G-8)

Have students rewrite mixed numerals such as the above in simpler form.

4. The numeral for a mixed number omits the plus sign to indicate addition in the numeral.
$(6\frac{2}{3} = 6 + \frac{2}{3})$

5. The problem of semantics and cumbersome language makes it impractical to always distinguish between number and numeral in many definitions and rules.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-3. Number names: Mixed numerals to fractions.	Obtain forms that are convenient for the operations of addition, subtraction, multiplication, and division.	Suggest to the student: We now know that $\frac{5}{2} = 2\frac{1}{2}$. Can we convert mixed numerals to fractions? $4\frac{2}{3} = \square$ We know: $4\frac{2}{3} = 4 + \frac{2}{3}$ What is a convenient fraction name for 4 so that you can find the sum in fraction form?  Convert the units of the number line into thirds. So $4\frac{2}{3} = \frac{12}{3} + \frac{2}{3}$ $= \frac{14}{3}$ Mathematical justification of this procedure is $4\frac{2}{3} = 4 + \frac{2}{3}$ $= (4 \times 1) + \frac{2}{3}$ $= (4 \times \frac{3}{3}) + \frac{2}{3}$ $= \frac{12}{3} + \frac{2}{3}$ $= \frac{14}{3}$	Mixed numerals can be renamed as fractions.
I-4. Operations. Addition of unlike fractions and mixed numbers	Build computational skills.	(1) $\frac{3}{4} + \frac{5}{8} = \square$ Illustrate on the number line. 	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-4 continued)		Can we add without using the number line? The only way is if they are like fractions. Convert to like fractions.	

$$\frac{3}{4} + \frac{5}{8} = \left(\frac{3}{4} \times 1\right) + \frac{5}{8}$$
$$= \left(\frac{3}{4} \times \frac{2}{2}\right) + \frac{5}{8}$$
$$= \left(\frac{3 \times 2}{4 \times 2}\right) + \frac{5}{8}$$
$$= \frac{6}{8} + \frac{5}{8}$$
$$= \frac{11}{8} \text{ or } 1\frac{3}{8}$$

(2) $\frac{2}{3} + \frac{3}{4} = \left(\frac{2}{3} \times 1\right) + \left(\frac{3}{4} \times 1\right)$

Raise the question — Which would be the most useful forms of 1? To get a common denominator we must get multiples of 3 and 4. (See F-13) The L.C.M. would be the simplest but not necessary.

L.C.M. $= 3 \times 4 = 12$

$$\frac{2}{3} + \frac{3}{4} = \left(\frac{2}{3} \times \frac{4}{4}\right) + \left(\frac{3}{4} \times \frac{3}{3}\right)$$
$$= \left(\frac{2 \times 4}{3 \times 4}\right) + \left(\frac{3 \times 3}{4 \times 3}\right)$$
$$= \frac{8}{12} + \frac{9}{12}$$
$$= \frac{17}{12} \text{ or } 1\frac{5}{12}$$

(3) $\frac{5}{12} + \frac{4}{9} = \frac{5}{2 \times 2 \times 3} + \frac{4}{3 \times 3}$

(L.C.M. $= 2 \times 2 \times 3 \times 3$)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-4 continued)		$\frac{5}{12} + \frac{4}{9} = (\frac{5}{2 \times 2 \times 3} \times \frac{3}{3}) + (\frac{4}{3 \times 3} \times \frac{2 \times 2}{2 \times 2})$ $= \frac{5 \times 3}{2 \times 2 \times 3 \times 3} + \frac{4 \times 2 \times 2}{3 \times 3 \times 2 \times 2}$ $= \frac{15}{36} + \frac{16}{36}$ $= \frac{31}{36}$ Note carefully the differences in the procedure and format used in the above examples. Vertical forms should also be introduced for purposes of computation. $\begin{array}{r} \frac{2}{3} \\ + \frac{3}{4} \\ \hline \end{array}$	
I-5. Addition of mixed numbers.	Build computational skills.	Use approaches as in I-4. (1) a) $4\frac{3}{4} + 1\frac{1}{5} = \frac{19}{4} + \frac{11}{5}$ $= (\frac{19}{4} \times 1) + (\frac{11}{5} \times 1)$ $= (\frac{19}{4} \times \frac{5}{5}) + (\frac{11}{5} \times \frac{4}{4})$ $= \frac{19 \times 5}{4 \times 5} + \frac{11 \times 4}{5 \times 4}$ $= \frac{95}{20} + \frac{44}{20}$ $= \frac{139}{20}$ or $6\frac{19}{20}$	Mixed numbers can be added by either (1) Renaming as fractions, or (2) Using the format of adding two-digit numbers.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-5 continued)		b) Using the structure developed in the horizontal form, introduce and give examples using the vertical form. $\begin{array}{r} 4\frac{3}{4} = \frac{19}{4} = \frac{95}{20} \\ + 2\frac{1}{5} = \frac{11}{5} = \frac{44}{20} \\ \hline \frac{139}{20} \text{ or } 6\frac{19}{20} \end{array}$ (2) Compare the similarities in adding two-digit numbers and mixed numbers. (See B-15) e.g. $\begin{array}{r} 23 = 20 + 3 \\ + 45 = 40 + 5 \\ \hline 60 + 8 \text{ or } 68 \end{array}$ $\begin{array}{r} 4\frac{3}{4} = 4 + \frac{3}{4} = 4 + \frac{15}{20} \\ + 2\frac{1}{5} = 2 + \frac{1}{5} = 2 + \frac{4}{20} \\ \hline 6 + \frac{19}{20} \text{ or } 6\frac{19}{20} \end{array}$ The latter method applies because the former algorithm was built on the same associative and commutative properties of addition. Addition of combinations of mixed numbers with whole numbers or fractions can be similarly developed.	
I-6. Subtraction of unlike fractions. Subtraction of mixed numbers.	Build computational skills.	(1) Use approaches similar to those developed for addition in I-4. Note that if problems are taken at random from students some subtractions are not meaningful. (2) Use approaches as developed in I-5 where regrouping is not necessary.	(1) Unlike fractions can be subtracted by renaming them as like fractions, provided the subtrahend is less than, or equal to, the minuend.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-6 continued)		(3) When re-grouping is necessary, such as $6\frac{1}{3} - 2\frac{3}{4}$, again compare with the two-digit example because they are built on the same properties. $\begin{array}{r} 43 = 40 + 3 = 30 + 13 \\ - \underline{27 = 20 + 7 = 20 + 7} \\ 10 + 6 \text{ or } 16 \end{array}$ $\begin{array}{r} 6\frac{1}{3} = 6 + \frac{1}{3} = 6 + \frac{4}{12} = 5 + \frac{16}{12} \\ - \underline{2\frac{3}{4} = 2 + \frac{3}{4} = 2 + \frac{9}{12} = 2 + \frac{9}{12}} \\ 3 + \frac{7}{12} \text{ or } 3\frac{7}{12} \end{array}$ Cite other examples such as $7 - 2\frac{1}{3}$ and $5\frac{2}{3} - 3$. (Have students note that re-grouping is not necessary in the second example.) Note also that if problems are performed by method (2a) from the SPECIFIC UNDERSTANDING FOR STUDENT, the regrouping would not be necessary.	(Fractions are not closed under subtraction.) (2) Mixed numbers can be subtracted by either a) Renaming as fractions, or b) Using the format of subtracting two-digit numbers. (3) In order to subtract, when the fractional part of the minuend is less than the fractional part of the subtrahend, the minuend must be renamed so that the fractional part in the minuend will be larger.
I-7. Addition and subtraction of measures.	Develop computational skills	Refer to examples of addition and subtraction of two-digit and mixed numbers to show that the structure properties of commutativity and associativity which were the bases of the algorithm apply equally to operations with measures. (1) Addition without re-grouping $\begin{array}{r} 23 = 20 + 3 \\ + \underline{42 = 40 + 2} \\ 60 + 5 = 65 \end{array}$	Addition and subtraction of measures are similar to that of two-digit ~~mixed~~ numbers. Numbers (measures) are added, not the units of measures.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-7 continued)			

$$\begin{array}{rl} & 2\text{ ft. } 3\text{ in.} = 2\text{ ft.} + 3\text{ in.} \\ + & 4\text{ ft. } 2\text{ in.} = 4\text{ ft.} + 2\text{ in.} \\ \hline & 6\text{ ft.} + 5\text{ in.} = 6\text{ ft. } 5\text{ in.} \end{array}$$

(2) Addition with re-grouping

$$\begin{array}{rl} & 2\frac{1}{4} = 2 + \frac{1}{4} \\ + & 1\frac{3}{4} = 1 + \frac{3}{4} \\ \hline & 3 + \frac{4}{4} = 3 + 1 = 4 \end{array}$$

$$\begin{array}{rl} & 2\text{ gal. } 1\text{ qt.} = 2\text{ gal.} + 1\text{ qt.} \\ + & 1\text{ gal. } 3\text{ qt.} = 1\text{ gal.} + 3\text{ qt.} \\ \hline & 3\text{ gal.} + 4\text{ qt.} = 4\text{ gal.} \end{array}$$

(3) Subtraction with re-grouping

$$\begin{array}{rl} & 81 = 80 + 1 = 70 + 11 \\ - & 12 = 10 + 2 = 10 + 2 \\ \hline & 60 + 9 = 69 \end{array}$$

$$\begin{array}{rl} & 8\text{ yd. } 1\text{ ft.} = 8\text{ yd.} + 1\text{ ft.} = 7\text{ yd.} + 4\text{ ft.} \\ - & 1\text{ yd. } 2\text{ ft.} = 1\text{ yd.} + 2\text{ ft.} = 1\text{ yd.} + 2\text{ ft.} \\ \hline & 6\text{ yd.} + 2\text{ ft.} = 6\text{ yd. } 2\text{ ft.} \end{array}$$

$$\begin{array}{rl} & 5 = 4 + \frac{10}{10} \\ - & 2\frac{1}{10} = 2 + \frac{1}{10} \\ \hline & 2 + \frac{9}{10} = 2\frac{9}{10} \end{array}$$

$$\begin{array}{rl} & 5\text{ m.} = 4\text{ m.} + 100\text{ cm.} \\ - & 2\text{ m. } 1\text{ cm.} = 2\text{ m.} + 1\text{ cm.} \\ \hline & 2\text{ m.} + 99\text{ cm.} = 2\text{ m. } 99\text{ cm.} \end{array}$$

Inter-relate selections of units of measure to those previously taught which need review and which compare favorably (gallons and quarts to fourths, metric system or dollars and cents to whole numbers). (See H-7)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-8. Multiplication of mixed numbers.	Build computational skills.	Seek suggestions from students for methods of solving problems such as the following. Refer to methods used in addition of mixed numbers for ideas where necessary. Example 1 — Example 2 (a) $2\frac{3}{4} \times 5\frac{1}{2}$ — $4\frac{2}{5} \times 5$ Rename as fractions: $\frac{11}{4} \times \frac{11}{2}$ — $\frac{22}{5} \times \frac{5}{1}$ Use previous methods for fractions. $\frac{11 \times 11}{4 \times 2}$ — $\frac{22 \times 5}{5 \times 1}$ $\frac{121}{8}$ — $\frac{22}{1} \times \frac{5}{5}$ $15\frac{1}{8}$ — $\frac{22}{1} \times 1$ $\frac{22}{1}$ 22 (b) $2\frac{3}{4} \times 5\frac{1}{2}$ Compare $(2 + \frac{3}{4}) \times (5 + \frac{1}{2})$ to $(20 + 3) \times (50 + 1)$ The same commutative, associative, distributive, and identity properties apply. Therefore the algorithm for 23×51 based on these applies to $2\frac{3}{4} \times 5\frac{1}{2}$. $\begin{array}{rcr} 23 & = & 20 + 3 \\ \times\ 51 & = & 50 + 1 \\ \hline 1 \times 3 & = & 3 \\ 1 \times 20 & = & 20 \\ 50 \times 3 & = & 150 \\ 50 \times 20 & = & 1000 \\ \hline & & 1173 \end{array}$	Mixed numbers may be multiplied by (a) Renaming as fractions, or (b) Using the format of multiplying two-digit numbers. The format of method (b) is possible, but not as simple as method (a).

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-8 continued)			

$$\begin{array}{rl} 2\frac{3}{4} & = 2 + \frac{3}{4} \\ \times\ 5\frac{1}{2} & = 5 + \frac{1}{2} \\ \hline \end{array}$$

$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8} = \frac{3}{8} = \frac{3}{8}$$
$$\frac{1}{2} \times 2 = 1 = 1 = 1$$
$$5 \times \frac{3}{4} = \frac{15}{4} = 3\frac{3}{4} = 3\frac{6}{8}$$
$$5 \times 2 = 10 = 10 = 10$$
$$\overline{15\frac{1}{8}}$$

$$\begin{array}{rl} 42 & = 40 + 2 \\ \times\ 5 & = 5 \\ \hline & 200 + 10 = 210 \end{array} \qquad \begin{array}{rl} 4\frac{2}{5} & = 4 + \frac{2}{5} \\ \times\ 5 & = 5 \\ \hline & 20 + 2 = 22 \end{array}$$

Going through all steps horizontally for an example such as $4\frac{2}{5} \times 5$ and reviewing the structural properties involved will check the validity of the algorithm and the fact that both parts are multiplied by 5.

$$\begin{aligned} 5 \times 4\tfrac{2}{5} &= 5 \times (4 + \tfrac{2}{5}) \\ &= (5 \times 4) + (5 \times \tfrac{2}{5}) \\ &= 20 + (\tfrac{5}{1} \times \tfrac{2}{5}) \\ &= 20 + (\tfrac{5 \times 2}{1 \times 5}) \\ &= 20 + (\tfrac{5}{5} \times \tfrac{2}{1}) \\ &= 20 + (1 \times 2) \\ &= 20 + 2 \\ &= 22 \end{aligned}$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-9. Multiplication of measures by rational numbers.	Develop computational skills.	1. Jack is making bookshelves. He needs 8 shelves each 3 ft. 5 in. long. How much lumber will he need? (See I-8)	Multiplication of measures is similar to multiplication of multi-digit numbers or mixed numbers. Measures are multiplied, not the units of measure.

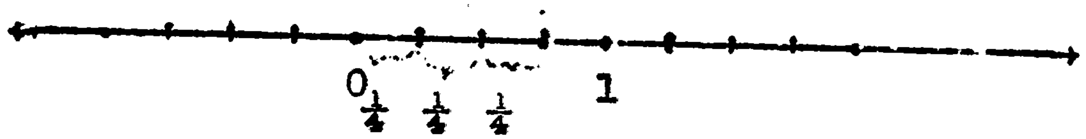
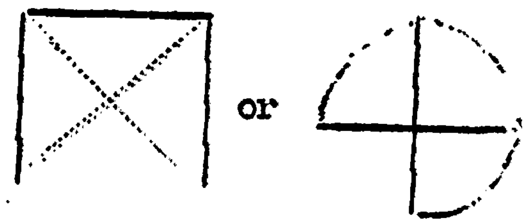
$$\begin{array}{rcl} & 3 \text{ ft. } 5 \text{ in.} = 3 \text{ ft.} + 5 \text{ in.} & \\ \times & 8 \quad = \quad 8 & \\ \hline & 24 \text{ ft.} + 40 \text{ in.} & = 24 \text{ ft.} + (3 \text{ ft.} + 4 \text{ in.}) \\ & & = (24 \text{ ft.} + 3 \text{ ft.}) + 4 \text{ in.} \\ & & = 27 \text{ ft.} + 4 \text{ in.} \\ & & = 27 \text{ ft. } 4 \text{ in.} \end{array}$$

An alternate method may be shown which would rename the measurements in the smallest unit being used. Rename the product in terms of the original or larger unit. This method can best be developed by reviewing and comparing to the method of renaming mixed numbers as fractions before multiplying.

$$\begin{array}{rl} & 3 \text{ ft. } 5 \text{ in.} = 41 \text{ in.} \\ \times & 8 \quad = \ 8 \\ \hline & 328 \text{ in.} = 27 \text{ ft. } 4 \text{ in.} \end{array}$$

$$\begin{aligned} 8 \times 3\tfrac{5}{12} &= 8 \times \tfrac{41}{12} \\ &= \tfrac{328}{12} \\ &= 27\tfrac{4}{12} \\ &= 27\tfrac{1}{3} \end{aligned}$$

2. Mary was making brownies. She wants to make half a recipe which calls for 1 cup 3 tablespoons flour. How much flour will she need?

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-9 continued)		$\frac{1}{2} \times (1 \text{ cup } 3 \text{ tbsp.}) = \frac{1}{2} \times (1 \text{ cup} + 3 \text{ tbsp.})$ $= (\frac{1}{2} \times 1 \text{ cup}) + (\frac{1}{2} \times 3 \text{ tbsp.})$ $= \frac{1}{2} \text{ cup} + \frac{3}{2} \text{ tbsp.}$ $= \frac{1}{2} \text{ cup} + 1\frac{1}{2} \text{ tbsp.}$	
I-10. Division of fractions.	Extend the division concept to fractions. Build division algorithms for fractions.	In view of the fact that fractions can be added, subtracted, and multiplied, can we divide them? $\frac{3}{4} \div \frac{1}{4} = \square$ To decide, we must consider what we mean by division. Using the number line or regions, find how many fourths are contained in three-fourths by counting congruent segments or regions. $\frac{3}{4}$   Is this procedure analagous to that for whole numbers? Suggest to students that they try more difficult examples such as $\frac{7}{4} \div \frac{1}{8}$ (different denominators), $\frac{7}{8} \div \frac{1}{4}$ (mixed number quotient), $\frac{1}{4} \div \frac{1}{2}$ (quotient less than one).	One fraction can be divided by another. <u>Any</u> fraction can be divided by another unless the divisor is a name for zero. The concept that division of whole numbers is counting the number of equivalent subsets applies similarly to division of fractions where we count the number of congruent segments or regions.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-10 continued)		Use carefully worded questions such as "How many fourths <u>of a unit</u> are contained in seven-eighths of the <u>same unit</u>?" Dropping the words "of a unit" may mislead a child to think $\frac{1}{4}$ of $\frac{7}{8}$ instead of $\frac{7}{8} \div \frac{1}{4}$. Students should not separate the $\frac{7}{8}$ into 4 parts, but rather separate the unit into 4 parts and count the number of such parts needed to cover the $\frac{7}{8}$. The difficulty of solving division problems by the above methods should motivate the need by students for a simple algorithm.	
I-11. Division of fractions.	Build division algorithms for fractions.	1. Ask students what numbers would make the following sentences true. $2 \times \square = 1$, $\frac{1}{3} \times \square = 1$, $\frac{3}{4} \times \square = 1$, $\frac{5}{1} \times \square = 1$, $\frac{a}{1} \times \square = 1$, $\frac{a}{b} \times \square = 1$ Is there always a number you can multiply by another number and obtain a product of one? Students should recognize that this is true for all cases except zero. Review that $a \div b$ can be written as $\frac{a}{b}$. Extend the meaning of fraction to include forms where the numerator and denominator are themselves fractions. Would this new form (called complex fraction) represent a number? Yes, because this could be verified by using the inverse operation.	Two numbers are called reciprocals if their product is one. The term fraction will be used to include forms where the numerator and denominator are themselves fractions. This form is called a complex fraction. Division of a number by another number is the same as multiplying the number by the reciprocal of the divisor.

(I-11 continued)

Thus $\frac{3}{4} \div \frac{7}{8} = \dfrac{\frac{3}{4}}{\frac{7}{8}}$. Lead the students to realize that the simplest division operation is division by one.

$$\dfrac{\frac{3}{4}}{\frac{7}{8}} = \dfrac{\frac{3}{4}}{\frac{7}{8}} \times 1$$

$$= \dfrac{\frac{3}{4}}{\frac{7}{8}} \times \dfrac{\frac{8}{7}}{\frac{8}{7}}$$

$$= \dfrac{\frac{3}{4} \times \frac{8}{7}}{\frac{7}{8} \times \frac{8}{7}}$$

$$= \dfrac{\frac{3}{4} \times \frac{8}{7}}{1}$$

$$= \frac{3}{4} \times \frac{8}{7}$$

$$= \frac{3 \times 8}{4 \times 7}$$

$$= \frac{24}{28} \text{ or } \frac{6}{7}$$

Alternate Approach: $\dfrac{\frac{3}{4}}{\frac{7}{8}}$

What is the source of difficulty? (The fact that the numerator and denominator are not whole numbers.)

What number can be multiplied times the above fraction without changing the value? (One.)
What form of one would be used? (A fraction.)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-11 continued)		What fractional form of one would eliminate the denominators in the two parts? (A fraction whose numerator is a multiple of both given denominators.) What is a multiple of both? (24) What is the lowest? Do we need the lowest? (NO.) Use the fractional form of one which uses the multiple.	

$$\frac{\frac{3}{4}}{\frac{7}{8}} = \frac{\frac{3}{4}}{\frac{7}{8}} \times 1$$

$$= \frac{\frac{3}{4}}{\frac{7}{8}} \times \frac{24}{24}$$

$$= \frac{\frac{3}{4} \times 24}{\frac{7}{8} \times 24}$$

$$= \frac{18}{21} \text{ or } \frac{6}{7}$$

2. If $\frac{a}{b}$ (where $\underline{b}$ is not zero) and $\frac{c}{d}$ (where $\underline{d}$ is not zero) are any two fractions, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \times 1$$

$$= \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}}$$

$$= \frac{\frac{a}{b} \times \frac{d}{c}}{1}$$

$$= \frac{a}{b} \times \frac{d}{c}$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-11 continued)		Note — $\underline{c}$ cannot be zero as this would make $\frac{a}{c}$ meaningless. The above approach to a generalization should be used with better students. However, for other students only specific examples should be used to demonstrate the above understandings.	
I-12. Division of mixed numbers.	Build computational algorithms.	This can be handled in the same way as multiplication of mixed numbers was developed, by renaming mixed numerals as fractions (See I-8), and using the division algorithm developed in I-10. $2\frac{2}{3} \div \frac{1}{6} = \frac{8}{3} \div \frac{1}{6}$ $= \frac{8}{3} \times \frac{6}{1}$ $= \frac{8 \times 6}{3 \times 1}$ $= \frac{48}{3}$ or 16 $3\frac{1}{4} \div 1\frac{2}{3} = \frac{13}{4} \div \frac{5}{3}$ $= \frac{13}{4} \times \frac{3}{5}$ $= \frac{13 \times 3}{4 \times 5}$ $= \frac{39}{20}$ or $1\frac{19}{20}$	Mixed numbers can be divided by renaming as fractions and making use of the reciprocal.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-13. Division of measures by rational numbers.	Build computational skills.	1. A board 9 ft. 6 in. long is to be divided into three boards of equal length. How long is each board? a) Since: $3\overline{)96}$ can be written as $3\overline{)90 + 6}$ Then: $3\overline{)9 \text{ ft. } 6 \text{ in.}}$ can be written as $\begin{array}{r} 3 \text{ ft.} + 2 \text{ in.} = 3 \text{ ft. } 2 \text{ in.} \\ 3\overline{)9 \text{ ft.} + 6 \text{ in.}} \\ \underline{9 \text{ ft.}} \\ 6 \text{ in.} \\ \underline{6 \text{ in.}} \end{array}$ b) $9 \text{ ft. } 6 \text{ in.} \div 3 = 9\frac{1}{2} \div 3$ $= \frac{19}{2} \div \frac{3}{1}$ $= \frac{19}{2} \times \frac{1}{3}$ $= \frac{19}{6}$ or $3\frac{1}{6}$ or 3 ft. 2 in. c) $9 \text{ ft. } 6 \text{ in.} \div 3 = 114 \text{ in.} \div 3$ $= \frac{114}{3}$ in. or 38 in. or 3 ft. 2 in. 2. $8 \text{ gal. } 2 \text{ qt.} \div 5 = \begin{array}{r} 1 \text{ gal.} + 2\frac{4}{5} \text{ qt.} \\ 5\overline{)8 \text{ gal.} + 2 \text{ qt.}} \\ \underline{5 \text{ gal.}} \\ 3 \text{ gal.} + 2 \text{ qt.} = 14 \text{ qt.} \\ \underline{10 \text{ qt.}} \\ 4 \text{ qt.} \end{array}$ Methods (1-b) and (1-c) may also be used.	Division of measures is similar to division of multi-digit numbers or mixed numbers. Measures are divided, not units of measure.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-14. Rational numbers. Properties.	Ascertain which properties for whole numbers are valid for fractions.	Review the properties of whole numbers. (See G-15) Ask students which apply to fractions. Include: Is there a smallest fractional number? What is the next larger fractional number after zero? Is there a largest fractional number? Is there a next larger fractional number? Can fractions be ordered? Is the sum of two fractions larger than either addend? Is the difference of two fractions less than the minuend? Can any two fractions be added? subtracted? multiplied? divided? Is the product of two fractions larger than the factors? Is the product of a fraction and zero equal to zero?	A few properties that are true for whole numbers are not true for fractions. (Example: The product may be smaller in value than the factors.) Fractions have certain properties that whole numbers lack. (Example: Closure for division except for division by zero.)
I-15. Decimals. Decimal numerals.	Introduce decimal numerals.	Review the concept of obtaining place values: a) A place value can be obtained by multiplying the place value on its right by ten. b) A place value can be obtained by dividing the place value on its left by ten.	Place value is used in decimal numerals. Decimal numerals can be written in expanded notation. Numerals expressed in expanded notation can be written as decimal numerals.

Ten-thousands	Thousands	Hundreds	Tens	Ones
10 × 1,000	10 × 100	10 × 10	10 × 1	1
100,000 ÷ 10	10,000 ÷ 10	1,000 ÷ 10	100 ÷ 10	10 ÷ 10

Using idea (b) extend the place values to the right of the ones place.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

I-15 continued)

Thousands	Hundreds	Tens	Ones				
10×100	10×10	10×1	1				
$10,000 \div 10$	$1,000 \div 10$	$100 \div 10$	$10 \div 10$	$1 \div 10$ or $\frac{1}{10}$	$\frac{1}{10} \div 10$ or $\frac{1}{100}$	$\frac{1}{100} \div 10$ or $\frac{1}{1,000}$	$\frac{1}{1,000} \div 10$ or $\frac{1}{10,000}$

A decimal point is used to help identify place value position in the numeral by placing the decimal point between the ones place and the tenths place.

Have students check to see if characteristic (a) is still true. What names can be given to the new place value positions in the above chart?

Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
$1000 \div 10$	$100 \div 10$	$10 \div 10$	$1 \div 10$	$\frac{1}{10} \div 10$	$\frac{1}{100} \div 10$
			$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

Using a chart with the same headings as the previous one, insert digits in several columns and ask the children how these numbers could be written in decimal numerals (without using the written place value names). This should motivate the need for the decimal point.

Hundreds	Tens	Ones	Tenths	Hundredths	
4	6	7			= ____
	4	6	7		= ____
		4	6	7	= ____

Provide activities of writing numerals in expanded notation:

$$234 = 200 + 30 + 4$$
$$= (2 \times 100) + (3 \times 10) + (4 \times 1)$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-15 continued)		$2.34 = 2 + .3 + .04$ $= (2 \times 1) + (3 \times \frac{1}{10}) + (4 \times \frac{1}{100})$ Provide activites of writing numerals expressed in expanded notation as decimal numerals: $(3 \times 10) + (4 \times 1) + (0 \times \frac{1}{10}) + (7 \times \frac{1}{100}) = 34.07$ Give students plenty of practice in reading and writing decimal numerals. Have students write numerals read by another student from a given list. Do students write the correct numeral? Is the communication vocabulary sufficient? Include examples such as 400.003 and .800 . Do the students write .403 and .00008 instead? Lead to the need to limit the use of "and" to mixed decimal numerals and the hyphen (read by inflection) for place value. Also accept the form where the decimal point is read "point." Thus 34.07 is read as "thirty-four and seven hundredths" or "thirty-four point zero seven."	
		In this guideline the terms "decimal" and "decimal numeral" will refer to names of numbers expressed in the base ten numeration system. Generally speaking, these terms will refer to numerals representing numbers less than one and mixed numbers. These numerals will normally have a decimal point in them to identify the ones place. However, it is recognized that a whole number is also usually named by a base ten numeral and thus is a "decimal." Whole numbers will usually be referred to as such in this guideline to try to better communicate with the reader. To avoid confusion in semantics the term "decimal" will be used to mean the number represented as well as the numeral form used to name it.	"Decimals" usually refer to forms in base ten of numbers less than one and mixed numbers. Whole numbers, while not included in common usage in the word "decimal," do meet the requirements of the definition.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-16. Decimal numbers. Conversion.	Prepare for multiplication of decimals.	Have students list fractions less than one whose denominators are multiples of ten. Have the students express their fractional numerals as decimal numerals. $\frac{3}{10} = .3$ $\frac{3}{100} = .03$ $\frac{3}{1,000} = .003$ $\frac{24}{100} = .24$ $\frac{24}{1,000} = .024$ $\frac{24}{10,000} = .0024$ Discuss the use of zero as a place holder to the right of the ones place. Repeat the same procedure for numbers equal to or greater than one expressed as fractions or mixed numerals. $\frac{63}{10} = 6\frac{3}{10} = 6.3$ $\frac{603}{100} = 6\frac{3}{100} = 6.03$ The students now should be able to express decimal numerals as mixed numerals or fractions whose denominators are multiples of ten. $.6 = \frac{6}{10}$ $.06 = \frac{6}{100}$ $.006 = \frac{6}{1,000}$ $.49 = \frac{49}{100}$ $.049 = \frac{49}{1,000}$ $4.27 = 4\frac{27}{100} = \frac{427}{100}$ $3.08 = 3\frac{8}{100} = \frac{308}{100}$	A mixed numeral or a fraction numeral having a denominator that is a multiple of ten can be expressed as a decimal numeral. A decimal numeral can be expressed as a fraction numeral having a denominator that is a multiple of ten. Zeros are often needed to show place value position to the right of the ones place in writing decimal numerals.
I-17. Decimal numbers. Addition and subtraction.	Extend the operations of addition and subtraction to decimals.	Ask the students to find the sum of 270 and 184 by using expanded notation. (See B-11, B-12, B-15, B-1) Ask the students if they can now find the sum of 2.7 and 1.84 by expanded notation.	The basic structure of place value and addition can be applied to addition of decimals.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-17 continued)		Approaches might b..	The basic structure of place value and subtraction can be applied to the subtraction of decimals. For ease of computation, like place value positions are aligned by aligning the decimal points in the numerals in the vertical form of addition and subtraction.

(1)

$$\begin{array}{rl} 2.7 & = (2 \times 1) + (7 \times \frac{1}{10}) \\ + 1.84 & = (1 \times 1) + (8 \times \frac{1}{10}) + (4 \times \frac{1}{100}) \\ \hline & = (3 \times 1) + (15 \times \frac{1}{10}) + (4 \times \frac{1}{100}) \\ & = (3 \times 1) + [(10 \times \frac{1}{10}) + (5 \times \frac{1}{10})] + (4 \times \frac{1}{100}) \\ & = (3 \times 1) + [1 + (5 \times \frac{1}{10})] + (4 \times \frac{1}{100}) \\ & = [(3 \times 1) + (1 \times 1)] + (5 \times \frac{1}{10}) + (4 \times \frac{1}{100}) \\ & = (4 \times 1) + (5 \times \frac{1}{10}) + (4 \times \frac{1}{100}) \\ & = 4 + .5 + .04 \\ & = 4.54 \end{array}$$

(2)

$$\begin{array}{rl} 2.7 & = 2 + \frac{7}{10} \\ + 1.84 & = 1 + \frac{8}{10} + \frac{4}{100} \\ \hline & = 3 + \frac{15}{10} + \frac{4}{100} \\ & = 3 + 1 + \frac{5}{10} + \frac{4}{100} \\ & = 4 + \frac{5}{10} + \frac{4}{100} \\ & = 4 + .5 + .04 \\ & = 4.54 \end{array}$$

(3)

$$\begin{aligned} 2.7 + 1.84 &= (27 \times .1) + (184 \times .01) \\ &= (270 \times .01) + (184 \times .01) \\ &= (270 + 184) \times .01 \\ &= 454 \times .01 \\ &= 4.54 \end{aligned}$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-17 continued)		(4) $2.7 = \frac{27}{10} = \frac{270}{100}$ $+ 1.84 = \frac{184}{100} = \frac{184}{100}$ $\frac{454}{100} = 4.54$ or in the convenient vertical form: $2.7 = 2.70$ $+ 1.84 = 1.84$ 4.54 (5) Have the students add $\frac{8}{10}$ and $\frac{5}{10}$ and ask them to express the answer in simplest form. $\frac{8}{10} + \frac{5}{10} = \frac{13}{10}$ $= 1\frac{3}{10}$ Now ask them to express the same addends in decimal form and find the sum. $.8 + .5 = 1.3$ Note that both sums express the same number. Ask the students to find the sum of 4.8 + 3.724 + 28.45 without using expanded notation. Show the advantage of vertical form for actual computation. The student should find that in order to obtain a correct sum, the digits in like place value position must be added. In this approach the child must name both addends in a form where a common factor is evident and then apply the distributive prpperty of multiplication over addition. The problem then becomes one of addition of whole numbers. The common factor determines the position of the units place in the sum.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-17 continued)		The students should discover that when the addends are written vertically and the digits in like place value positions are lined up, the decimal point in each numeral is also aligned with the point in each of the other numerals. The alignment of the decimal point is a convenient guide in listing decimal numerals to find the sum. The development of subtraction of decimals can be taught using the same methods as in addition. Special attention should be given to regrouping situations, such as: a) $\begin{array}{r} 6.96 \\ -\ 2.48 \\ \hline \end{array}$ b) $\begin{array}{r} 3.95 \\ -\ 1.8 \\ \hline \end{array}$ c) $\begin{array}{r} 4.62 \\ -\ 2.375 \\ \hline \end{array}$	
I-18. Decimal numbers. Multiplication.	Extend the operation of multiplication to decimals.	Review multiplication of mixed numbers, (See I-8) stressing denominators which are powers of ten. $(\frac{1}{10} \times \frac{1}{100} = \frac{1}{1,000})$ Ask the students ways of multiplying 2×4.3. Possible responses include: (1) $\begin{array}{r} 4.3 \\ +\ 4.3 \\ \hline 8.6 \end{array}$ (repeated addition) (2) $2 \times 4\frac{3}{10} = \frac{2}{1} \times \frac{43}{10} = \frac{86}{10} = 8\frac{6}{10} = 8.6$ (3) $\begin{array}{l} 4.3 = (4 \times 1) + (3 \times \frac{1}{10}) \\ \underline{\times\ 2 = \qquad\qquad\qquad \times\ 2} \\ \quad (8 \times 1) + (6 \times \frac{1}{10}) \end{array}$ or $8\frac{6}{10}$ or 8.6	The basic structure and properties underlying place value and multiplication of whole numbers can be applied to multiplication of decimal numbers.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(I-18 continued)

(4) $$\begin{array}{r} 4.3 = 43 \times \frac{1}{10} \\ \times 2 = \quad \times 2 \\ \hline 86 \times \frac{1}{10} \text{ or } 43 \times \frac{2}{10} \end{array}$$

either of which can be renamed as $\frac{86}{10}$ or 8.6

If students obtain the product of $86 \times \frac{2}{10}$ in (4)

by multiplying 2 times each factor in the multiplicand, compare answer (3) with (4). What is the basic difference? What structural properties apply? Since both + and × are involved in (3), the distributive property of multiplication over addition applies. Since only × occurs in (4), the associative and commutative properties of multiplication are used and 2 is multiplied times <u>one</u> other factor.

In the vertical method, two approaches could be:

(1) Since
$$3.2 \times 2.48 = \frac{32}{10} \times \frac{248}{100} = \frac{32 \times 248}{1,000} = \frac{7936}{1,000} = 7.936$$
therefore
$$\begin{array}{r} 2.48 \\ \times\ 3.2 \\ \hline 496 \\ 7\ 44\ \ \\ \hline 7.936 \end{array}$$
(Traditional)

(2) Since
$$3.2 \times 2.48 = (3 + \frac{2}{10}) \times \frac{248}{100}$$
$$= (3 \times \frac{248}{100}) + (\frac{2}{10} \times \frac{248}{100})$$

by distributive law of multiplication over addition, therefore

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(I-18 continued)

$$\begin{array}{r} 2.48 \\ \times\ 3.2 \\ \hline .496 \\ 7.44 \\ \hline 7.936 \end{array}$$

$.496$ because $\frac{2}{10} \times \frac{248}{100} = \frac{496}{1,000}$

7.44 because $\frac{3}{1} \times \frac{248}{100} = \frac{744}{100}$

(Decimal points are kept aligned throughout.)

Using a variety of examples and recording the products in decimal form on a chart similar to the one below, help the students discover the generalization that the total number of digits to the right of the ones place in the two factors is the number of digits to the right of the ones place in the product.

factors	product	total number of digits to the right of the decimal point in 2 factors.	number of digits to the right of the decimal point in the product
.24, 2.3	.552	3	3
.24, 27.	6.48	2	2
.3, .25	.075*	3	3

* Stress the position and need for the zero in the product.

The total number of digits to the right of the ones place in two factors expressed in decimal form, is equal to the number of digits to the right of the ones place in the product.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-19. Decimal numbers. Division.	Extend the operation of division to decimals.	Ask the students to find the quotient in problems having a whole number divisor and a decimal dividend. Express the quotient as a decimal. Suggest problems whose remainder is zero. Approach 1 Example 1 $4.2 \div 7 = \square$ $\square \times 7 = 4.2$ Since $.6 \times 7 = 4.2$ (.6 obtained by trial and error) $\square = .6$ $7\overline{)\,4.2}$ quotient $.6$ Approach 2. Another way to show that .6 is the correct quotient would be: $4.2 \div 7 = \frac{42}{10} \div \frac{7}{1}$ $= \frac{42}{10} \times \frac{1}{7}$ $= \frac{42}{70}$ $= \frac{6}{10}$ $= .6$ Example 2 $18.36 \div 6 = \square$ $\square \times 6 = 18.36$ $3.06 \times 6 = 18.36$ $\square = 3.06$ $6\overline{)\,18.36}$ quotient 3.06	Decimals can be divided by whole numbers. When dividing a decimal dividend by a whole number, it is necessary to express the remainder after each subtraction in terms of the next smaller place value position.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-19 continued)		Since 3.06 is not likely to be discovered by trial and error, it could be derived by a method such as the following and then inserted back.	

$$18.36 \div 6 = \frac{1836}{100} \div \frac{6}{1}$$
$$= \frac{1836}{100} \times \frac{1}{6}$$
$$= \frac{1836 \times 1}{100 \times 6}$$
$$= \frac{1836 \times 1}{6 \times 100}$$
$$= \frac{1836}{6} \times \frac{1}{100}$$
$$= 306 \times \frac{1}{100}$$
$$= 3.06$$

Approach 3 (Distributive Property)

Review the distributive property used in division. (See E-23) Will the distributive property used in division of whole numbers hold for decimals?

(1) Without regrouping

$$8.4 \div 4 = (8 + .4) \div 4$$
$$= (8 \div 4) + (.4 \div 4)$$
$$= 2 + .1$$
$$= 2.1$$

(2) With regrouping

$$6.4 \div 4 = (6 + .4) \div 4$$
$$= (6 \div 4) + (.4 \div 4)$$
$$= 1 + (2 \div 4) + (.4 \div 4)$$
$$= 1 + (2.4 \div 4)$$
$$= 1 + .6$$
$$= 1.6$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(I-19 continued)

When this concept is understood, the <u>short</u> <u>form</u> can be used:

```
    2.146
7 ) 15.025
    14
     1 0   (tenths)
       7
       32  (hundredths)
       28
        45 (thousandths)
        42
         3
```

Another approach can be to change the mixed number to thousandths immediately:

(a)

```
    2.146
7 ) 15.025 = 15,025 thousandths  |
             14,000              |  2.
              1,025 thousandths  |
                700              |  .1
                325 thousandths  |
                280              |  .04
                 45 thousandths  |
                 42              |  .006
                  3 thousandths  |
```

(b)

```
    2 + .146 = 2.146
7 ) 15.025 = 15 + 25 thousandths
    14
     1 = 1000 thousandths
         + 25 thousandths          146
         1025 thousandths     7 ) 1025
```

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-19 continued)			

To motivate dividing 2 by 4 where the dividend is smaller than the divisor and there is no whole number quotient, ask, "Can the 2 be further partioned and distributed among the 4 groups?" Yes, by regrouping the 2 into smaller pieces (tenths).

$$\begin{array}{rl}
 & 1.6 \\
4 & \overline{)\,6.4} \\
 & \underline{4} \\
 & 2 = 20 \text{ tenths} \\
 & \underline{+\ 4 \text{ tenths}} \\
 & 24 \text{ tenths} \\
 & \underline{-24} = 4 \times .6 \\
 & 0
\end{array}$$

By extending to regrouping more than once:

$$\begin{array}{l}
\quad 2 + .1 + .04 + .006 = 2.146 \\
7\overline{)\,15.025} = 15 + 0 \text{ tenths} + 2 \text{ hundredths} + 5 \text{ thousandths} \\
\quad \underline{14} \\
\quad 1 = 10 \text{ tenths} \\
\quad \underline{+\ 0 \text{ tenths}} \\
\quad 10 \text{ tenths} \\
\quad \underline{-\ 7 \text{ tenths}} = 7 \times .1 \\
\quad 3 \text{ tenths} = 30 \text{ hundredths} \\
\quad \underline{+\ 2 \text{ hundredths}} \\
\quad 32 \text{ hundredths} \\
\quad \underline{-\ 28 \text{ hundredths}} = 7 \times .04 \\
\quad 4 \text{ hundredths} = 40 \text{ thousandths} \\
\quad \underline{+\ 5 \text{ thousandths}} \\
\quad 45 \text{ thousandths} \\
\quad \underline{-\ 42 \text{ thousandths}} = 7 \times .06 \\
\quad 3 \text{ thousandths}
\end{array}$$

Use of transparencies with overlays on an overhead projector may be particularly useful on involved algorithms as in the above.

(I-19 continued)

(c)

```
7) 15.025
   14.000 | 2.
    1.025 |
     .700 |  .1
     .325 |
     .280 |  .04
     .045 |
     .042 | .006
     .003   2.146
```

What do you do with the remainder?

Possibility 1

Write it in the quotient as a remainder.

Example:
```
      2.146 r .003
   7) 15.025
```

Possibility 2

What would the effect be on the last digit in the quotient (.006)? Can we divide the remainder by the divisor and find whether the result is more or less than one half of the last place value in the quotient?

Method 1

3 thousandths $\div$ 7 = $\frac{3}{7}$ thousandths which is less than half of .001 .

Method 2

Rename .003 as .0030 and divide by 7 to obtain .0004 .

Since in either case the quotient is less than half of .001, the quotient is closer to 2.146 than to 2.147 . If it were larger than half of .001, the quotiant would would be closer to 2.147 than to 2.146 . If it were exactly half of .001, ther is no good reason to choose either one of 2.146 or 2.147 but common practice dictates choosing the larger.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-20. Decimals. Devision.	Express fractions in decimals.	Is 23 = 23.0 ? Is 23 = 23.00 ? Is 23 = 23.000 ? Is 2.4 = 2.40 ? Is 2.4 = 2.400000 ? Is .04 = .0400 ? What generalizations can be made about the number represented when zeros are placed in the place value positions to the right of the last digit in any given decimal numeral?	Any number of zeros may be placed in the place value positions to the right of the last digit in a given decimal numeral without changing its value. With whole numbers, identify the ones place by inserting the decimal point before annexing zeros.
		Can $\frac{3}{4}$ be expressed as a decimal numeral? How can $\frac{3}{4}$ be interpreted? (1) $\frac{3}{4} = 3 \div 4$ $\begin{array}{l} \quad \underline{0 + .7 + .05} = .75 \\ 4\overline{)3} \\ \quad \underline{0} \\ \quad 3 = 30 \text{ tenths} \\ \qquad \underline{28} \text{ tenths } (4 \times .7) \\ \qquad 2 \text{ tenths} = 20 \text{ hundredths} \\ \qquad\qquad \underline{20} \text{ hundredths } (4 \times .05) \\ \qquad\qquad 0 \end{array}$	Any number expressed as a fraction can be expressed as a decimal numeral. Every fraction with a whole number numerator and counting number denominator can be written as a terminating decimal or a repeating decimal.
		A decimal quotient in which a remainder of zero is obtained is a "terminating decimal." (2) Do all decimal quotients terminate? Try others such as $\frac{1}{7}$: $\frac{1}{7} = 1 \div 7$	A decimal quotient with a remainder of zero is termed a "terminating decimal." A non-terminating quotient in which the same digits re-occur in the same order is termed a "repeating decimal."

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-20 continued)		$$\begin{array}{r} 0.14285714 \\ 7\overline{)1.00000000} \\ \underline{0} \\ 1\,0 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$	The recurring digits in a repeating decimal are expressed with a bar above them. A rational number is a number which can be expressed as (1) a fraction with a whole number numerator and a counting number denominator and also as (2) a repeating decimal or terminating decimal.
		Will we ever get a remainder or zero? When should we stop? How many possible remainders can be obtained? (Seven.) This means the remainder must repeat after no more than seven partial quotients. From that point, the computation will repeat itself. After discussion a general agreement should be reached that the instructions for each example should indicate the last place value position desired in the quotient or until it repeats or terminates.	
		(3) $\frac{8}{3} = 8 \div 3$	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-20 continued)		$$\begin{array}{r} 2.666 \\ 3\overline{)8.000} \\ \underline{6} \\ 2\,0 \\ \underline{1\,8} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$ **.666.... is called a repeating decimal and is written** $.\overline{6}$, so $\frac{8}{3} = 2.\overline{6}$. In example (2), $\frac{1}{7} = .\overline{142857}$. Discuss and define rational number which includes both fractional and decimal forms of the same numbers.	
I-21. Decimal numbers. Division.	Extend division of decimals to include a decimal divisor.	Ask the students if they can find the quotient of (1) $$16.8 \div 2.4 = 16\frac{8}{10} \div 2\frac{4}{10}$$ $$= \frac{168}{10} \div \frac{24}{10}$$ $$= \frac{168}{10} \times \frac{10}{24}$$ $$= \frac{168}{24}$$ Does **this** method always work? (Yes!) Is it a convenient method? (Not always!) Is there an easier method? Let's look.	Decimals can be divided by decimals. In a division problem with a decimal divisor it is convenient to express the problem in an equivalent form in which the divisor is a whole number

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-21 continued)		(2) $2.4\overline{)16.8}$ with quotient $.7$. If this response is given, ask the student to check his answer using multiplication. Using this method, is there a convenient method of determining the place value of the quotient? (NO!) Therefore this method is not useful.	A decimal that is not a whole number can be expressed as a whole number by multiplying the decimal by a multiple of ten.
		(3) $2.4\overline{)16.8}$ with quotient $7.$: 16.8 − 12.0 = 4.8 (5); 4.8 − 4.8 = 0 (2); total 7. $1.2\overline{)1.728}$ with quotient 1.44: 1.728 − 1.200 = .528 (1); .528 − .480 = .048 (.4); .048 − .048 = 0 (.04); total 1.44.	Dividing a decimal by a decimal can also be accomplished by methods such as: (1) repeated subtraction (2) converting to mixed fraction forms and dividing.

The student can divide a decimal by what kind of divisors? How can the original problem be expressed as a problem having a whole number divisor? (By multiplying the divisor and dividend by the same multiple of ten.) To most students this is more evident by expressing the problem in fractional form and multiplying by a fractional name for one.

$$\left(\frac{\text{multiple of ten}}{\text{multiple of ten}}\right)$$

$$1.2\overline{)1.728} = \frac{1.728}{1.2} = \frac{1.728}{1.2} \times 1 = \frac{1.728}{1.2} \times \frac{10}{10} = \frac{1.728 \times 10}{1.2 \times 10} = \frac{17.28}{12} = 12\overline{)17.28}$$

a form with which the student is already familiar and which is equivalent to the original problem.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(I-21 continued)		Encourage students to omit steps to arrive directly to $1.2\overline{)1.728} = 12\overline{)17.28}$ If students insist on not writing the problem in the equivalent form, they will need to indicate the positions of the unit's place in the equivalent problem. Since additional decimal points would be confusing, carots are sometimes used but are not necessary.	
I-22. Verbal problems.	Build skills and confidence in interpreting verbal problems.	Verbal problems have been used as motivation and as sources of examples in teaching computational algorithms with fractions and decimals. Now present a series of lessons where emphasis is on relating environmental situations to writing and interpretation of mathematical sentences. Include situations that are normally encountered but not written out as verbal problems. (Buying and selling for decimals, building a garage or sewing a dress for fractions and measurement.) Avoid use of isolated "clue" words as the sole determinant of the operation involved, since situations originate in the environment where "clues" do not exist and since "clue" words may be misleading. Initiate attacking such problems by determining specifically the information requested and by evaluating the information available as a basis of determining possible useful operations. Emphasize analysis of the problem in terms of the concepts behind operations (partitioning, union, ordered arrays, comparison, separation). Students should discover that finding numbers which make the mathematical sentence true are sometimes best found by using an inverse operation: $\square \div \frac{2}{3} = 9$ Rewrite as $\frac{2}{3} \times 9 = \square$	Environmental situations can be interpreted using mathematical sentences where the numbers which make the sentence true can be easily determined.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
I-23. Percent. Introduction.	Build background for later problems in business and science.	What other number names occur in newspapers which are unfamiliar? (6%, 25%, $5\frac{1}{2}\%$) Indicate that % is a name for "$\times \frac{1}{100}$." Have students rename numerals in percent forms as (1) fractional forms by multiplying, and as (2) decimal forms by renaming directly, or renaming as a fraction and dividing.	% is a name for "$\times \frac{1}{100}$" Numbers written in per-cent forms can be renamed as fractions. Numbers written in per-cent forms can be conveniently renamed as decimals by first renaming as fractions.

Fraction

$$6\% = 6 \times \frac{1}{100} = \frac{6}{1} \times \frac{1}{100} = \frac{6}{100}$$

$$5\tfrac{1}{2}\% = 5\tfrac{1}{2} \times \frac{1}{100} = \frac{11}{2} \times \frac{1}{100} \text{ or } \frac{5\frac{1}{2}}{100} = \frac{11}{200} \text{ or } \frac{5\frac{1}{2}}{100}$$

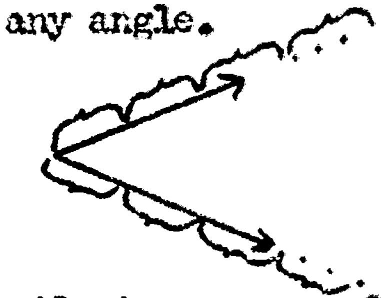
$$4.2\% = 4.2 \times \frac{1}{100} = \frac{42}{10} \times \frac{1}{100} = \frac{42}{1,000}$$

Decimal

$$6\% = 6 \times \frac{1}{100} = \frac{6}{100} = 100\overline{)6} = .06 \quad \text{or } 6\% = 6 \times \frac{1}{100} = 6 \times .01 = .06$$

$$5\tfrac{1}{2}\% = \frac{11}{200} = 11\overline{)200} = .055 \quad \text{or } 5\tfrac{1}{2}\% = 5\tfrac{1}{2} \times \frac{1}{100} = 5.5 \times .01 = .055$$

$$4.2\% = \frac{42}{1,000} = 1,000\overline{)4.2} = .042 \quad \text{or } 4.2\% = 4.2 \times \frac{1}{100} = 4.2 \times .01 = .042$$

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-1. Angles and angular regions.	Develop concept and ability to measure "angles" or directions.	Review unit segment, unit square region, and the need for a basic unit similar to the thing being measured.	Using rays to measure angles would not be useful because every angle has a measure of two rays.
		1. Ask students for possible units to measure any given angle and consider each suggestion. Segment? It would take an endless number to measure any angle. 	A unit angular region is used to measure an angular system. In common usage, the words "measuring angles" really denotes measuring the angular region.
		Rays? Exactly two rays would "cover" any angle. Degrees? What is a degree? Does it represent a set of points? (Teacher, See J-4) Angle? How would you place and count unit angles to cover the angle and only the angle being measured? Protractor? Is this a unit or a tool like a ruler? (A tool.) To distinguish one angle from another such as  do we really think of comparing the measures of angles or their regions?	Only congruent angles will cover one another. Therefore, unit angles are of no value in measuring angles.
		2. Consider measuring an angular region. What unit would you use? Segment? Linear units can only be used to measure segments. Unit of area such as square inch? An endless number would be needed.	

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-1 continued)			

In what way are angular regions different from other regions which could be measured with units such as the square inch? Note that an angular region is not bounded by a closed curve as other regions are.

Ray? An endless number are needed.

Must not the unit and that which is to be measured be similar? (Yes.) What then must the unit be?

An angular region? (Yes.)

What size? (Any size.)

How would you place them to cover the region and to get the minimum number needed?

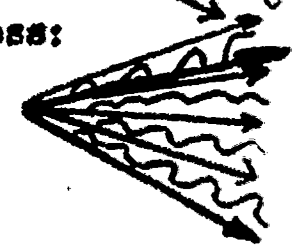
Try an example and experiment.

Note the kind of unit needed to cover a region that does not end. (Answer: a unit that does not end.)

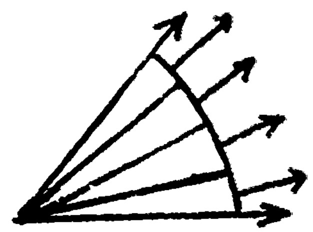
Region to be measured:

Unit region:

Process:



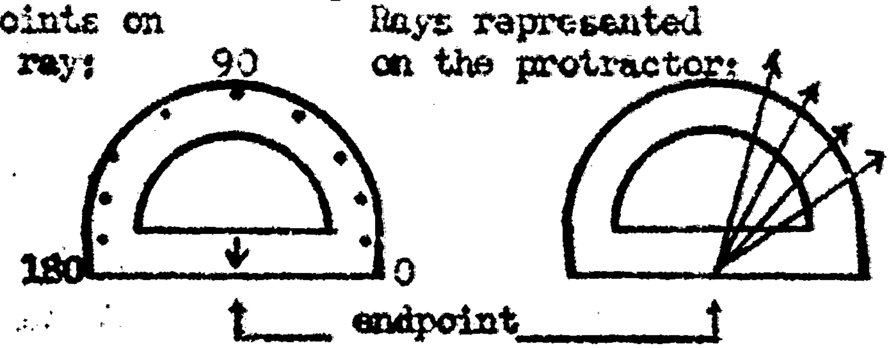
Give students practice by having each student cut out arbitrary representations of a unit angle from construction paper and count the number of his units needed to cover some given angular region. The student must be aware that his model regions actually represent unending regions.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-1 continued)		 Discuss the fact that measuring angular regions is usually simply denoted as measuring angles.	
J-2. Standard unit.	Introduce the standard unit-degree.	Raise questions: Is there a need for a standard unit? Why? What is a standard unit? (The degree.) How could you define a unit so that others will understand and how "big" should it be? Define degree. Discuss that the number needed to cover the plane is strictly arbitrary or relate historically to its source in Babylon where a number base of 60 was used. Introduce degree symbolism. Could there be other standard units? Do you think there are other standard units? There is another "standard unit" called the radian.	The unit standard angular region is the degree. A degree is an angular region such that the union of 360 such regions covers the plane. The symbol for degree is a raised circel such as 30°.
J-3. Use of the protractor.	Familiarize student with common measuring tools and build skills in usage.	Have students cut out a unit angular region. How many are needed to cover the plane? Repeat with a smaller unit. How small must the unit be to get a degree (i.e., 360 fill the plane)? Could an instrument be devised to show the number of unit angular regions needed to cover a given angular region? What must be shown on the instrument? (Answer: Rays which form angular regions.) How much of the ray must be shown? (Answer: The endpoint and another point on the ray.)	The degree is too small to conveniently manipulate models. The protractor is a convenient tool for measuring angular regions. m $\angle ABC$ is a symbol meaning the measure of angle ABC, and represents a number.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(J-3 continued)

Show a protractor and point out the above features. *



Introduce the needs for proper usage such as lining up the common endpoint (vertex) and the ray bounding the region with the mark on the protractor indicating the first ray (usually marked 0).

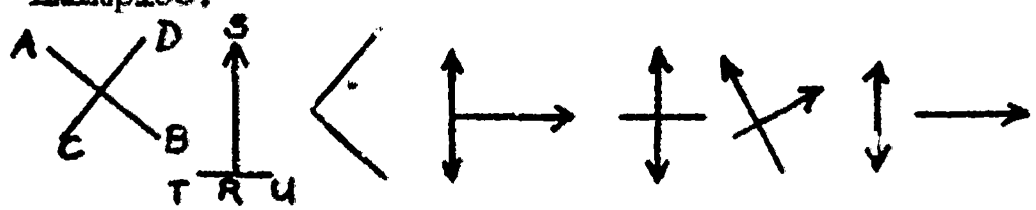
Give students practice in
(1) measuring given angular regions, and
(2) constructing angular regions with a given measure.

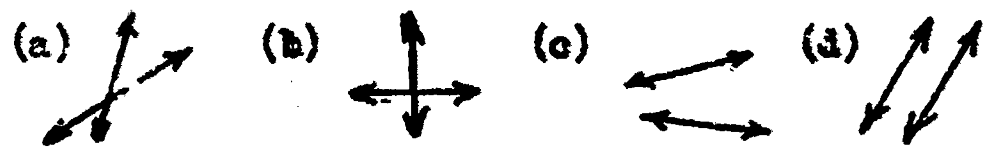
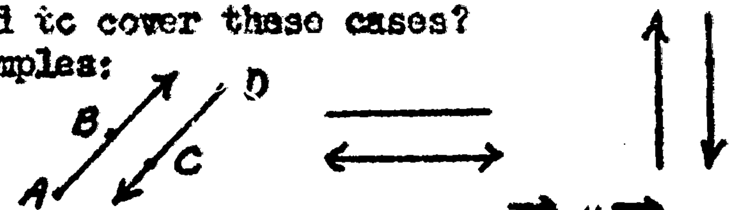
Introduce symbolism such as m $\angle$ABC (in degrees) = 30 or the measurement of $\angle$ABC is approximately 30°.

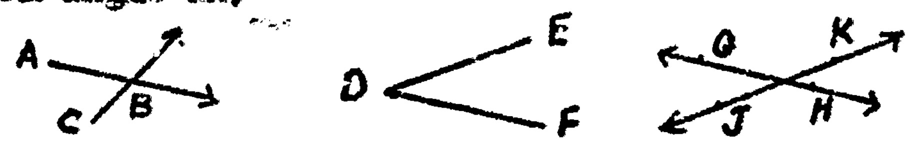
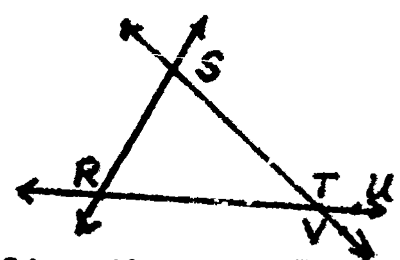
Ask what is wrong with $\angle$RST = 45°.
(Answer: The left side represents a set of points. The right side represents a measurement. These are not names for the same idea and an equal sign (=) should not be used.

* If the question of a circular protractor arises, point out that these instruments exist but that there is no need since every interior region is covered with the semicircular instrument.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-4. Angle names	Classify angles. Introduce common terminology.	Represent many angles and have students group together those which they think are similar. Discuss the common characteristics and assign a name.	A right angle is an angle whose angular region has a measurement of 90°.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-4 continued)		(Many texts define a straight line as a straight angle. This guide does not because the definition of an angle excludes a straight line for the reason that such an "angle" has no interior region to measure to obtain 180°.)	An acute angle is an angle whose angular region has a measurement less than 90°. An obtuse angle is an angle whose angular region has a measurement more than 90°.
J-5. Perpendicularity.	Introduce the concept and definition of perpendicularity, a characteristic found in many geometric figures.	1. Refer to a representation of a right angle. What other common terminology is used to describe the relationship of the rays?	1. Each of two rays in a plane which form a right angle is perpendicular to the other.
		2. Have students draw pairs of intersecting lines. Do any of the examples illustrate the same idea of perpendicularity? How could you define or determine which ones are perpendicular?	2. Each of two lines is perpendicular if two rays from the point of intersection are perpendicular.
		3. What other geometric figures can be perpendicular? How would you define or determine whether they are perpendicular? Examples: 	3. Any combination of lines, rays, and segments are perpendicular if the lines containing these figures are perpendicular.
		4. Name the rays, lines, and segments and introduce the symbol $\perp$ in describing their relationship: $\overline{AB} \perp \overline{CD}$ $\overrightarrow{RS} \perp \overline{TU}$	4. $\perp$ is the symbol for "is perpendicular to."

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-6. Parallel concept.	Analyze characteristics of and derive definitions of many common geometric figures.	Have students represent many pairs of lines in a plane. Are there any cases in which they do not intersect? Examples:  Do the lines in (c) intersect? Yes!! (See D-13 and F-7) Is there a name for the relationship when two lines do not intersect? Yes — PARALLEL. Can other figures be parallel (such as rays and segments)? What definition for parallel could be used to cover these cases? Examples:  Introduce symbolism such as $\overrightarrow{AB} \parallel \overrightarrow{DC}$	Two lines are parallel to each other if and only if they lie in the same plane and do not intersect. Any combinations of rays, lines, and segments are parallel to each other if and only if the lines containing them are parallel. $\parallel$ is the symbol for "is parallel to."
J-7. Geometric figures. Triangles.	Introduce and define common closed geometric figures. Classify triangles.	Have some students represent triangles or trinagular regions by cutting paper and other students by building them from straws and pipe cleaners. Collect all models and break into subsets (or classes). Discuss the common characteristics of the subsets. Name and define the subsets. Collect the models again and repeat the above procedure, dividing into other possible subsets. Possibility 1: Triangles vs. triangular regions. Possibility 2: Large triangles vs. small triangles.	The classes of triangles, by lengths of sides, are: equilateral, isosceles, and scalene. The segments, the union of which form a triangle, are called sides. Equilateral triangles are triangles with 3 congruent sides. Isosceles triangles are triangles with 2 or more congruent sides.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-7 continued)		Possibility 3: Equilateral vs. isosceles vs. scalene. Possibility 4: Acute vs. right vs. obtuse.	Scalene triangles are triangles with no congruent sides.
		When angle relations are discussed, ask for definition of angle and question whether angles are represented. Show that angles are "determined" by extending the segments to form rays. Discuss the meaning of "determined" and "formed."	A figure which is "determin by an original figure inclu points in addition to those represented on the original figure or only part of those represented on the original figure.
		Do $\overrightarrow{AB}$ and $\overrightarrow{CH}$, $\overline{DE}$ and $\overline{DF}$, $\overleftrightarrow{GH}$ and $\overleftrightarrow{JK}$ form or determine an angle in: 	A figure which is "formed" includes all points represented on the original figur
		When extending the segments in a triangle, many (more than three) angles are determined. Examples: $\angle RST$, $\angle SRT$, $\angle TSR$, $\angle UTV$, $\angle STU$ 	An interior angle of a trian is an angle determined by th two rays which are extension of 2 sides of the triangle, and in which the angular region contains the interior of the triangle.
		But triangle implies three angles! How can a particular three be determined? (Answer: By considering only interior angles.) But what should an interior angle be?	The classes of triangles, by angles, are: acute, right, and obtuse.
		When classifying triangles by number of right angles or obtuse angles, why was not a separate subset for the triangle with two right or two obtuse angles considered? (Answer: There are noneas can be seen by trying to draw such figures.)	Acute triangles are triangla in which all interior angles are acute. Right triangles are triangle in which one interior angle is a right angle. Obtuse triangles are triangl in which one interior angle obtuse. A triangle can determine onl one right or obtuse interior angle.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

TOPICAL OUTLINE

J-8.
Geometric figures.

Quadrilaterals.

PURPOSE OR OBJECTIVE

Classify quadrilaterals.

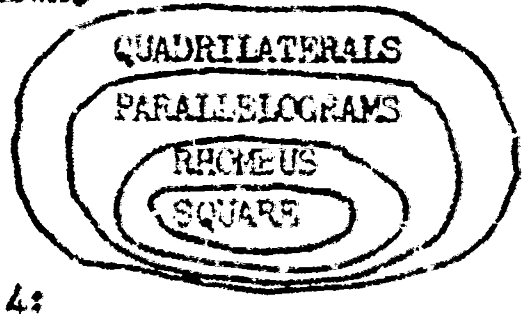
TEACHING APPROACH AND EXAMPLES

Repeat TEACHING APPROACH in J-7, but with quadrilaterals.

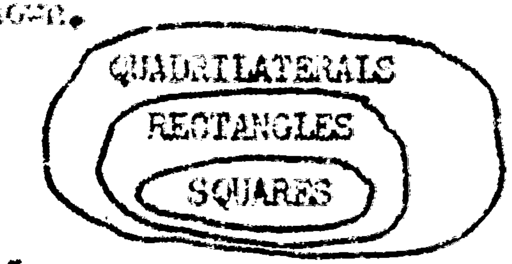
Possibility 1:
Quadrilateral vs. quadrilateral regions.

Possibility 2:
Large vs. small.

Possibility 3:
Figures with pairs of congruent sides vs. those without.
Further sub-divide and name the subsets, where known.



Possibility 4:
Figures determining right angles vs. those that do not.
Further sub-divide and rename subsets, where known.



Possibility 5:
Figures with pairs of parallel sides vs. those without.
Further sub-divide and rename subsets.

QUADRILATERALS
TRAPEZOIDS
PARALLELOGRAMS
RHOMBUS RECTANGLES
SQUARES

SPECIFIC UNDERSTANDING FOR STUDENT

The segments, the union of which form a polygon, are sides.

A parallelogram is a quadrilatera in which opposite pairs of sides are congruent. (By possibility 3)

A parallelogram is a quadrilatera with 2 pairs of parallel sides.

A trapezoid is a quadrilateral with at least one pair of paralle sides (and by some definitions only one pair of parallel sides).

A rhombus is a parallelogram in which all 4 sides are congruent to each other.

A square is a rhombus, the interi angles of which are right angles.

A rectangle is a parallelogram, the interior angles of which are right angles.

A square is a rectangle with 4 congruent sides.

The interior angle of a polygon i an angle determined by two rays which are extensions of two adjacent sides, in which the angular region contains the interior of the polygon.

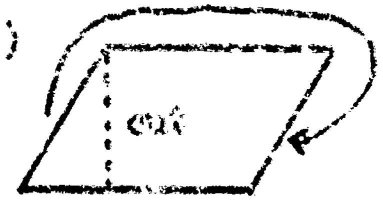
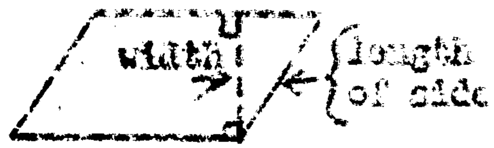
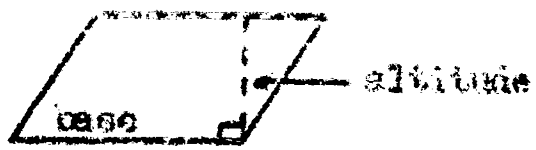
Adjacent sides are sides with a common endpoint.

Geometric figures are usually bes defined as naming them as a subse of another figure and listing the further restrictions.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-8 continued)		Using alternate definition of trapezoid, QUADRILATERALS TRAPEZOIDS PARALLELOGRAMS RHOMBUS RECTANGLES SQUARES	
J-9. Geometric figures. Polygons.	Classify polygons.	Repeat TEACHING APPROACH in J-7 with polygons, in general. Students will possibly sub-divide by numbers of sides. Name the common subsets. Another possibility is regular polygons vs. non-regular.	A polygon with exactly 3 sides is a triangle. A polygon with exactly 4 sides is a quadrilateral, not necessarily a square. A polygon with exactly 5 sides is a pentagon. A polygon with exactly 6 sides is a hexagon. A polygon with exactly 8 sides is an octagon. Polygons with a number of sides different from those named above have been assigned names which are less commonly used. Polygons in which all sides are congruent and all interior angles are congruent are called regular polygons.
J-10. Geometric figures. Polygons.	Compare geometry to world around us.	Have students review the names of the geometric figures and list applications where such figures (as polygons or regions) are found.	The geometric figures are commonly found in natural and man-made phenomena.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-11. Geometric figures. Properties.	Explore properties of figures not obvious from the definition.	Have students explore the relations between the interior angles by measuring and comparing, or tearing regions and physically comparing, in such figures as: parallelograms rectangles isosceles triangles equilateral triangles Have them also explore the parallel and perpendicular relationships between sides.	The opposite interior angles of parallelograms are congruent. The interior angles opposite the congruent sides of an isosceles triangle are congruent. The three interior angles of an equilateral triangle are congruent. The opposite sides of a parallelogram are congruent. The sum of the measures of the interior angles of any triangle are 180°. The sum of the measures of the interior angles of a quadrilateral are 360°.
J-12. Vocabulary. Diagonal.	Introduce common vocabulary. Prepare for formula for area of triangular regions.	Using models of quadrilaterals, pentagons, and hexagons, ask students what other segments, other than sides, are suggested. Draw them. Introduce the name diagonal and derive an appropriate definition. How many diagonals can be drawn in each figure?	A vertex is the point of intersection of two sides of a polygon. A diagonal is a segment which joins two vertices which are not endpoints of the same side of the polygon.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-13. Perimeter. Specialized formula.	Introduce special formulas which may involve less computation.	Give students a variety of problems on perimeter with intructions to compute by any formula or short cut that they can devise. Discuss alternate approaches and summarize as formulas. Some students will probably multiply rather than add. Emphasize that such specialized formulas are not necessary since the formula, $p = a + b + c + d$, will cover all cases with the appropriate number of sides.	Alternate or specialized formulas to determine perimeters for particular figures can be derived and are frequently used, such as: p (of square) = $4s$, where s is the measure of a side. p (of rectangle) = $2L + 2W$
J-14. Area. Formulas. Square.	Make students aware of common alternate formulas.	Ask students to determine the area of a floor tile by measuring and calculating. If students are "stuck" review that squares are rectangles, hence square regions are rectangular regions, and the formula for rectangular regions can be used. Since both measures (length and width) are the same, two separate letters are not necessary for the formula and the letter "s" is frequently used to denote measure of a side.	The measure of the square region "area" can be determined by using the formula for area of a rectangle or the special formula, $A = s \times s$, in which "s" is the measure of a side. The formula is alternately written as $A = ss$, $A = s \cdot s$, or $A = s^2$.
J-15. Areas. Formulas. Parallel-ograms.	Introduce common formulas. Prepare for formula for area of triangular regions.	Using models of parallelogram regions (including square, rectangular, and non-rectangular forms) ask students to measure and compute the area. (Some with non-rectangular figures may measure the lengths of both sides and multiply the measures.) Have students cut up the models and rearrange to form rectangular regions.	In a formula, capital A is used for measures of regions (area) and small letters are used for measures of segments. The bases of a parallelogram are any two sides which are not adjacent.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-15 continued)	Prepare for calculation of area of irregular figures.	(1)  (2)  Have other students measure and compute the area of rectangular regions formed by the first group after rearranging the figures. Are the predicted areas correct? Review the models that were cut up and rearranged until students note the difference between the length of a side and width.  Introduce terms, altitude and base, for the new formula as a reminder to avoid measuring the two adjacent sides.  How does determining area with a formula compare to counting congruent regions that will cover the original figure? (Answer: The latter is direct measurement. The former is indirect measurement in that a different quantity (length) is measured and a logical implication is drawn as to another quantity (area).	The altitude of a parallelogram is the segment with endpoints in each of the bases and which is perpendicular to the bases. The measure of the parallelogram region (area) is equal to the product of the measures of the base and altitude. ($A = bh$) Area formulas are based on indirect measurement. Indirect measurement is the determination of a measure by physically measuring another similar or dissimilar quantity and inferring a resulting measure of the original quantity.
J-16. Area. Formulas. Triangular regions.	Facilitate computing area of triangular regions.	Either (1) Have students draw a diagonal on models of parallelogram regions, cut, and compare the regions formed as to congruency, or (2) Combine congruent triangular regions to form parallelogram regions.	A parallelogram region can be divided into 2 congruent triangular regions. The base of a triangle is any one of the sides.

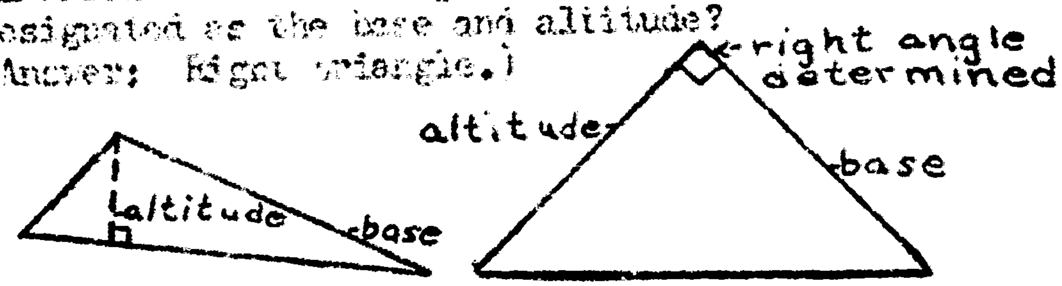
TOPICAL OUTLINE | PURPOSE OR OBJECTIVE | TEACHING APPROACH AND EXAMPLES | SPECIFIC UNDERSTANDING FOR STUDENT

(J-16 continued)

Teaching approach and examples:

Name the figures formed. (Answer: Triangular regions) How does the area of the triangular region compare to the area of the parallelogram region? What would be an appropriate formula? $A = \frac{1}{2}ab$

In which kind of triangle could two of the sides be designated as the base and altitude? (Answer: Right triangle.)



Give students practice using the formulas.

Specific understanding for student:

The altitude of a triangle is a segment which is perpendicular to the base with one endpoint in the base and the other endpoint coinciding with the vertex which is not in the base.

The area of a triangular region equals $\frac{1}{2}$ the area of a parallelogram region whose base and altitude are congruent to the base and altitude of the triangular region.

A (of triangular region) = $\frac{1}{2}ab$

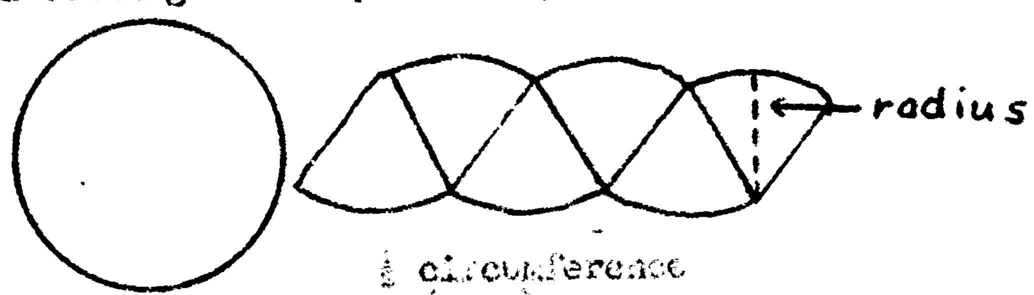
In a right triangle, the two short sides can be considered as the base and altitude.

J-17. Area. Formulas. Circular regions.

Purpose or objective: Measuring areas of circular regions.

Teaching approach and examples:

Give students models of congruent circular regions. Using a protractor, have students draw radii to partition the region into 6, 10 or 20 congruent portions, respectively. Cut out, like pieces of pie. Rearrange each student's parts to approximate a rectangular region. Does it more closely resemble a rectangular or parallelogram region?



$\frac{1}{2}$ circumference
$\frac{1}{2}$ $(2\pi r)$

Specific understanding for student:

The measure of a circular region (area) is determined indirectly by the product of π and two factors of the measure of the radius.

The formula for area of a circular region is $A = \pi rr$, where r is the measure of the radius or $A = \pi r^2$.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

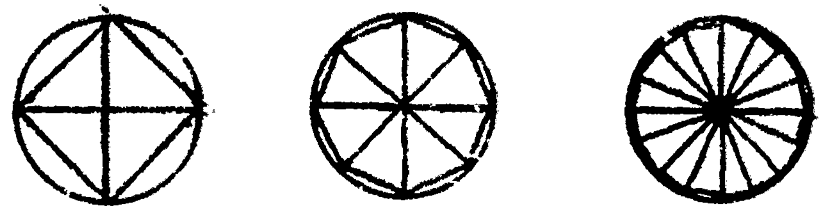
(J-17 continued)

What formula could be used to determine the area? How long is the altitude? (Answer: The measure of a radius.) How long is the base? (Answer: (One half circumference.) What is a formula for the measure of the circumference? ($C = 2\pi r$, See H-12) Derive a formula for the area of a circular region:

$$A = b \cdot a$$
$$A = (\tfrac{1}{2} \cdot c)r$$
$$A = \tfrac{1}{2}(2 \cdot \pi \cdot r)r$$
$$A = (\tfrac{1}{2} \cdot 2)\pi \cdot r \cdot r$$
$$A = 1 \cdot \pi \cdot r \cdot r$$
$$A = \pi \cdot r \cdot r$$

Alternate Approach

Demonstrate charts showing polygons of increasing number of sides inscribed in a circle.



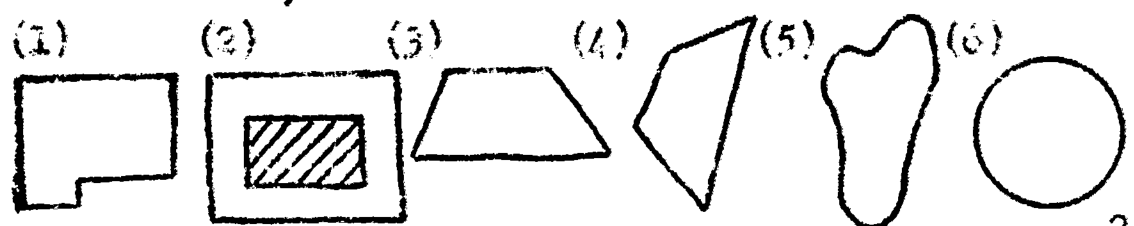
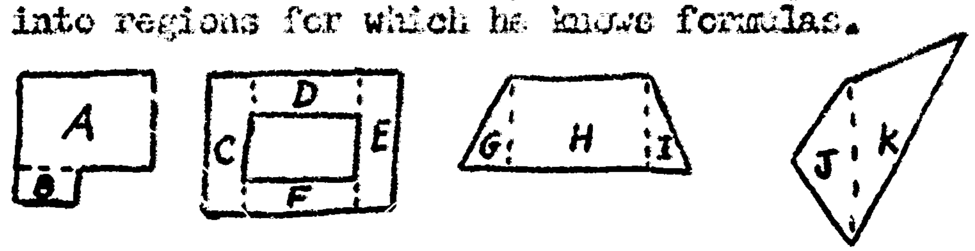
In each give a formula for the area of the polygon region by using the formula for the area of one of the congruent triangles formed. Commute and re-associate.

$A = 4(\frac{1}{2} ba)$	$A = 8(\frac{1}{2} ba)$	$A = 16(\frac{1}{2} ba)$
$A = \frac{1}{2}(4b)a$	$A = \frac{1}{2}(8b)a$	$A = \frac{1}{2}(16b)a$

or in general $A = \frac{1}{2} \cdot \text{perimeter} \cdot a$.
Since a = measure of a radius, $A = \frac{1}{2} \cdot p \cdot r$.

As the number of sides increases, length of p becomes closer and closer to a circumference, the formula for which is $c = 2\pi r$. Therefore, substituting:

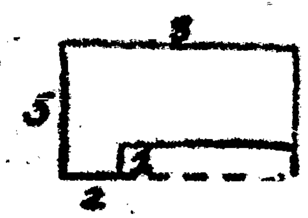
$A = \frac{1}{2}(2\pi r)r$
$A = \pi r^2$ by steps in part 1.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-18. Pi. Value.	Introduce alternate forms for computation.	Review E-12. Have students convert the $\frac{22}{7}$ to a mixed number and also to decimal form. $$7\overline{)22.0000}\quad 3.1428$$ Emphasize that 3.1428 is not correct to four decimal places since $\frac{22}{7}$ was only approximate. More accurate measurement of the circumference and diameter would lead to a quotient of 3.1416 to the closest ten-thousandths. Emphasize that it is not a terminating or repeating decimal. (Confirmed by computer methods) When would the decimal value be used rather than the fractional value?	(pi) has a value of approximately $\frac{22}{7}$, $3\frac{1}{7}$, or 3.1416. 3.1416 is more precise than $3\frac{1}{7}$.
J-19. Area. Irregular figures.	Introduce method for determining area when formulas are not available.	Introduce irregular regions where the areas are to be determined, such as:  Let students experiment and arrive at methods on their own. Guide, if necessary. Cases 1, 2, 3, 4. Some students will probably discover that the area can best be determined by partitioning the region into regions for which he knows formulas. 	1. To determine the area, most polygon regions can be conveniently partitioned into rectangular regions and triangular regions, and the areas of the parts added. 2. In some regions the area is best determined by finding the difference of two areas. 3. When other methods fail, estimation of area through counting blocks in a grid containing the figure can be used for an approximation.

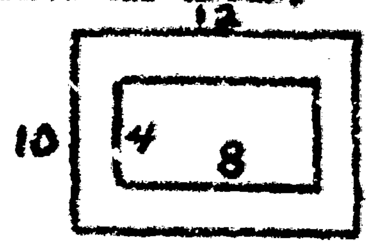
(J-19 continued)

Cases 1, 2.

In some cases, it is easier to extend segments to form larger regions and subtract the areas.



$(7 \times 5) - (1 \times 6)$



$(10 \times 12) - (4 \times 8)$

Cases 4, 5, 6, and 1, 2, 3 when dimensions are not known.

Trace the figure on to a grid (graph paper).

Method 1:

Count the blocks contained completely in the interior. Count the blocks needed to completely contain the region. Establish the possible range of measures. How can the range be narrowed?

Method 2:

Count all the blocks in which at least half is contained by the region.

Example:

Method 1: $15 < \text{area} < 31$
(Read $15 < \text{area}$ and $\text{area} < 31$)

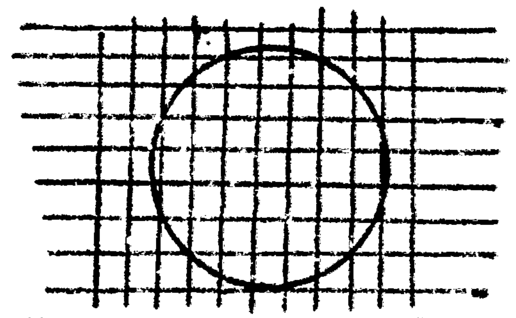
Method 2: 15

In method 2 what danger is encountered? (Answer: Cumulative error of parts not included may be appreciative. The maximum error is uncertain.)

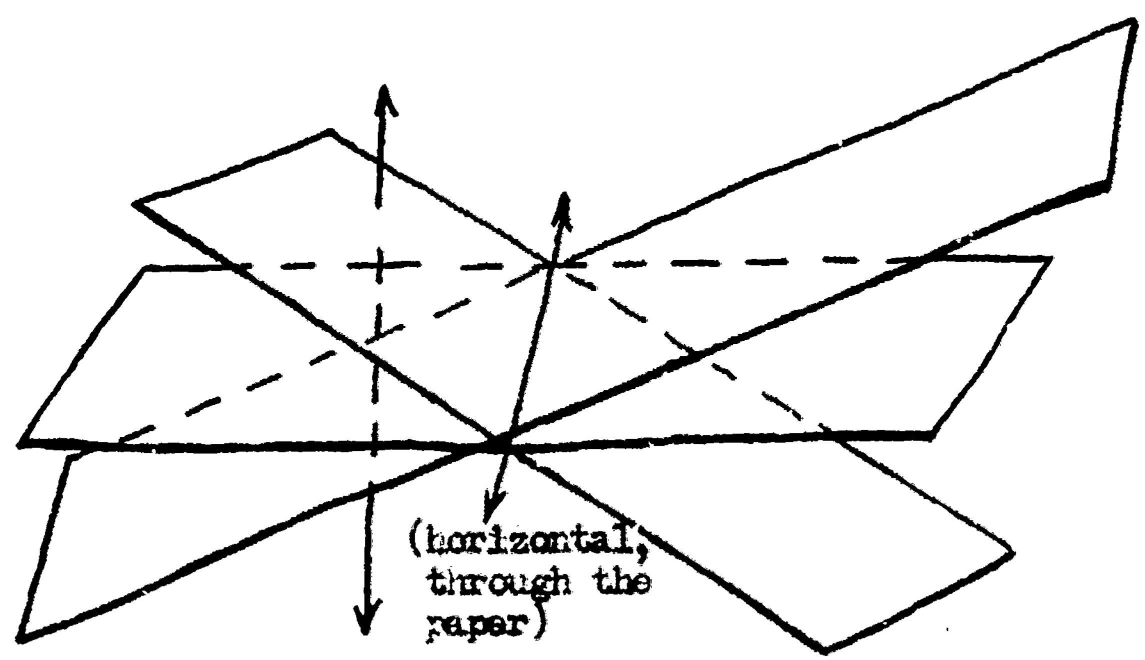
In either case, is the error within one unit as measurement normally implies? (NO)

As a third method, Method 2 can be used together with a correction based on estimation.

For a more accurate measurement, what would you use? (Answer: A grid with smaller squares.)

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-19 continued)		Experiment to see if the range of possible measures is less than that for larger squares. (Probably more.) Is the measure more accurate? (Yes because the unit is smaller.) Case 6. To check the value of in the formula $A = r$r, divide the area (in square inches) by the square of the measure of the radius (in inches). How close to the value of pi is this experimental result? Since $A = r^2$, $\frac{A}{r^2} =$ by the inverse relationship or the definition of division. 	
J-20. Geometry. Three-dimensions. Parallel.	Prepare for definitions of three-dimensional figures.	Review F-19 and J-6. Discuss: When are 2 lines parallel? Similarly, when are 2 planes parallel? When are a plane and line parallel? When are 3 planes or 3 lines (not in the same plane) parallel? Can the definition for 2 parallel lines be used in three dimensions? (NO) Refer to concrete models for discussion such as the floor and ceiling, three telephone poles, or a flag pole and wall.	Lines and/or planes are parallel if they do not intersect anywhere in space. Two regions are parallel is the two planes which contain the regions are parallel.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-21. Three-dimensions. Perpendicular.	Prepare for definitions of right prisms and right cylinders.	Using a stick and paper, have students demonstrate and attempt to define when a line and plane are perpendicular. Since new definitions must relate to old ones, which old ones are brought to mind? (Perpendicular lines.) How can a line perpendicular to a plane be related to perpendicular lines? Introduce new lines by drawing lines in the plane which intersect the original line. Now arrive at a definition. Are the top edge of the chalkboard and the vertical edge of the outside window perpendicular? Some students will undoubtedly feel that two non-intersecting lines can be perpendicular because a plane containing one of the lines is perpendicular to the other line. Use a model represented by the following drawing to show that many planes contain which are not perpendicular to the second line. Therefore, a notion of perpendicularity would be difficult to convey and is not used.	Two lines in space must intersect and must determine a right angle in order to be perpendicular. A line and a plane are perpendicular if the line and every line in the plane which passes through the intersection are perpendicular. Two planes are perpendicular if some line in one plane is perpendicular to the other plane.



TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-21 continued)		The indicated lines are not perpendicular. Why not? (They do not intersect.) Using a model such as two pieces of cardboard, demonstrate perpendicular planes. Arrive at a definition by relating to perpendicular lines or a perpendicular line and plane. Experiment by drawing representations for lines on the cardboard when students suggest lines.	
J-22. "Solid" geometric figures.	Introduce names for common geometric figures. Prepare for volume.	Review F-18. Use a set of many varied geometric models and also common models such as tin cans, boxes, pencils, ice cream cones (including various hollow models). Ask students to divide the models into appropriate classes. Introduce names such as face, vertex, and edge as necessary for discussion purposes, and define the name. Discuss the common characteristics of a subset; name and define the subset. Further subdivide each subset into new subsets where appropriate. Collect again and form new subsets by different common characteristics. Possibility 1: "Solids" vs. "surfaces" Possibility 2: Large vs. small Possibility 3: Closed surfaces all planar surfaces: prisms (cubes), pyramids some non-planar surfaces: cylinders, cones	Each of the plane surfaces that form a closed surface is a face. A segment which is the intersection of two faces is an edge. A point which is the intersection of three or more faces is a vertex. A closed surface in which all surfaces forming it are flat (planar) is a polyhedron. A prism is a closed surface in which (1) 2 polygon surfaces (called bases) are parallel, (2) the 2 polygon surfaces are congruent, and (3) all other surfaces are parallelogram regions joining the bases. Prisms are further classified as triangular, square, etc., depending on the shape of the base.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT

(J-22 continued)

Draw analogies with planar figures such as:

closed surface and closed curve

polyhedron and polygon

prisms and parallelograms.

Possibility 4:

Closed surfaces

cylinders — cones

prisms — pyramids

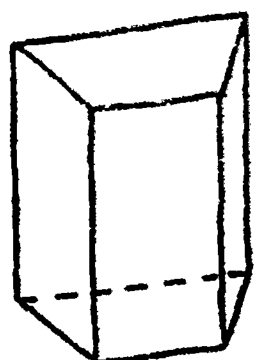
Possibility 5:

Closed surfaces

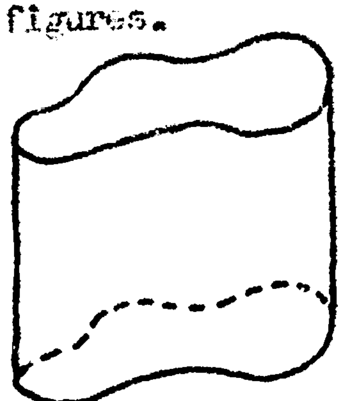
right surfaces — oblique surfaces

If students can enumerate the characteristics of a subset but cannot verbalize a formal definition, do not attempt to make students memorize the definitions, but test with questions such as "Is a pyramid a polyhedron?"

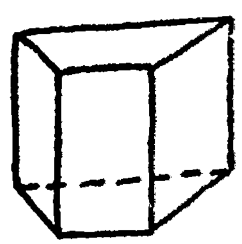
After working with concrete objects, students may wish to try drawing such figures.



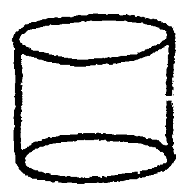
prism



cylinder



right prism



right circular cylinder

Prisms in which the parallelogram regions are rectangles are right prisms.

Cubes are prisms in which all surfaces are congruent square regions.

A cylinder is a closed surface in which (1) 2 regions (called bases) are parallel, (2) the bases are congruent, (3) the bases are regions which are bounded by simple closed curves and (4) sides are parallel segments whose endpoints are corresponding points of the simple closed curves.

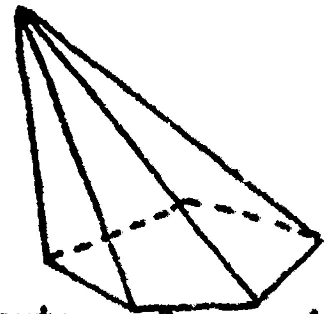
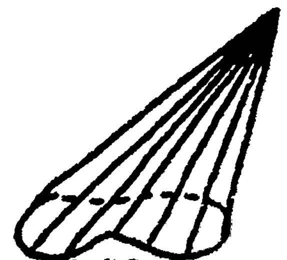
The lateral surface of a closed surface is all the closed surface exclusive of the base(s).

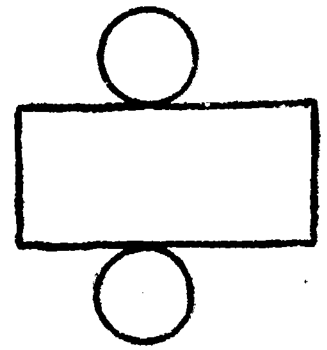
A circular cylinder is a cylinder whose base is a circular region.

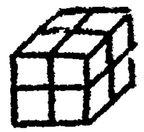
A right cylinder is a cylinder in which the parallel segments composing the lateral surface are perpendicular to the base.

A prism is one type of cylinder.

A pyramid is a closed surface consisting of (1) a polygon region (called base) and (2) triangular regions determined by an edge of the base and a common point not in the base (called apex).

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-22 continued)		 pentagonal pyramid  cone	Pyramids are further classified as triangular, square, etc., by the "shape" of the base. A cone is a closed surface consisting of (1) a region (called a base) which is bounded by a simple closed curve and (2) a surface consisting of segments one of whose endpoints is a point on the simple closed curve and whose other endpoint is a common point not in the base (called apex). A pyramid is one type of cone. A cone whose base is a circular region is a circular cone.
J-23. Measurement. Surface areas.	Prepare student for applications involving the amount of material needed to construct "hollow" figures.	Ask questions such as "How much cardboard is needed to build a box?" or "How much tin is needed to build a can?" Use hollow models that can be flattened. Discuss the common planar figures contained in the flattened pattern. What is meant by surface area? What kind of unit is needed to measure surface area? How could the area best be calculated? (Answer: Compute the area of each portion and add.) Have students experiment with models of right circular cylinders to discover that the surface area can be related to two circular regions and a rectangular region when cut and flattened.	The surface area of a figure is determined by calculating the area of each region which is part of the surface and adding the respective areas.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-23 continued)		 What is the length of the rectangle? (A circumference) Could the area of each point be computed? (Yes.)	
J-24. Volume. Concept.	Build concept of volume.	Review D-33. Using hollow figures and units such as one inch cubes, determine the measure of the volume. Would spherical units be better or worse? Could square units be used to measure the space region? In figures such as cylinders, what limits the accuracy of the result? (Answer: Spaces not filled with cubes.) Devise a method that would better fill the container such as filling a cubic inch container with sand, rice, or water, pouring into the container to be measured, repeating, and counting the units poured in.	The measure of a volume is the number of "solid" units such as cubes needed to fill the space region to be measured. Units made of material such as rice or water that take the shape of the container without changing the space occupied by the unit volume give a more accurate measure of volume than rigid units of measure. Units such as balls do not completely fill the space regions to be measured.
J-25. Volume. Standard units.	Introduce units needed for volume formulas.	Review the idea of needing a standard unit. Review some standard units of volume. (See D-21)	Some standard units of volume are based on units of length. Use cubes whose edges have a length of one standard unit of length.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
(J-25 continued)		What other standard units can be developed which are related to other measures such as length. Since blocks are frequently used as a unit, what size blocks would be most convenient? (Answer: Cubes with edges measuring one inch, one foot, one yard, one meter, one centimeter.) Build or show such unit models. What is the difference between a "1 inch cube" and "1 cubic inch?" (Answer: No difference in volume.) Between a "2 inch cube" and "2 cubic inches?"  2 inch cube  2 cubic inches Using models, build the relationships between various units such as: 1728 cu. in. = 1 cu. ft. 9 cu. ft. = 1 cu. yd. 1000 cu. cm. = 1 cu. m.	A 2 inch cube is a cube all edges of which have a length of 2 inches. Two cubic inches means 2 units, each of which is one cubic inch. Some common relations between standard volume units are: 1 cu. ft. = 1728 cu. in. 1 cu. yd. = 9 cu. ft. 1 cu. m. = 1,000 cu. cm.
J-26. Volume.	Develop formulas for volume.	Using "hollow containers" have students determine a systematic method of determining the number of one-inch cubes needed to fill the container (without counting each block). (If layers are formed, he need only count the number of blocks in a layer and number the layers.) Lead to the formula V = Bh, where B is the number of blocks in a layer (such as on the base) and h is the number of layers. Are there easy ways of counting blocks? (The area of a base would indicate the number of square units on which blocks could be placed to form a layer.) Are ther easy ways of determining the number of layers? (The measure of the height.) Discuss the formula V = Bh, where B could not denote the area of the base and h, the measure of the height.	The volume of space regions formed by cylinder and prism is determined by multiplying the area of the base by the measure of the height. The height of a cylinder or prism is the segment which (1) has one endpoint contained in each base, and (2) is perpendicular to the base. The measure of the height indicates the number of layers of units in the space region.

TOPICAL OUTLINE	PURPOSE OF OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDINGS FOR STUDENT
(J-26 continued)		In a rectangular prism (box), how could the number of blocks in a layer be determined without completely filling the layer? (Answer: Find the number of blocks needed for a row along adjacent edges.) Through measurement of lengths, the measures of the length and width can be found and area of the base computed. ($B = lw$) Lead from $V = Bh$ to $V = lwh$ for the volume of the space region. Using the model of a circular cylinder, could the formula $V = Bh$ be useful for finding the volume of the space region? How could the $\underline{B}$ be determined? (Answer: Since the base is a circular region, the area formula is $A = \pi r^2$ and πr^2 can be substituted for $\underline{B}$ in the volume formula.)	The area of the base indicates the number of units in a layer. The formula for volume of a cylinder or prism is $V = Bh$. The formula for volume of a rectangular solid is $V = lwh$. The formula for volume of a circular cylinder is $V = r^2h$. Volume of a closed surface is the common terminology meaning volume of the interior space region which is bounded by the surface. Formulas for volume are useful only if units of volume related to length are used.
J-27. Volume. Cones and pyramids.	Relate volume of cones and pyramids to cylinders and prisms.	Use models in which the cone and cylinder (and pyramid and prism) have congruent bases and heights. Have students experiment by filling the containers with rice, sand, or water and pouring from one container to another. If students wish to develop the formulas for volume of a cone and pyramid, this should be allowed but not encouraged.	The volume of a cone is $\frac{1}{3}$ the volume of a cylinder in which the base and height are congruent to those in the cone. The volume of a pyramid is $\frac{1}{3}$ the volume of a prism in which the base and height are congruent to those in the pyramid. The height of a cone or pyramid is the segment (1) in which the endpoints are the apex and a point in the base, and (2) which is perpendicular to the base.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-28. Measurement. Weight, time, and others.	Generalize concept of measurement to include all cases.	What is measured other than length, area and volume? (Answer: Time, temperature, weight, mass, speed, intensity of light and sound.) Do the same ideas of measurement apply, such as the need for a unit, a unit similar to the quantity to be measured, counting the number of units needed to "cover" that which is measured? Yes. "Covering," however, needs a different implication to include ideas such as time or temperature. Discuss the properties a unit must have (kind of unit) to be useful for each case. Discuss what "size" units would be convenient standard units. Discuss how "size" is frequently related to natural phenomena.	The principle of measurement of length, area, and volume apply to measurement of any quantity, whether tangible or not. Standard units (such as day and year) are related to natural phenomena (such as earth rotation, revolution). A unit mass (weight) must be related to a given volume of a given material found in nature.
J-29. Standard units.	Familiarize students with common standard units.	Have students name familiar standard units for measuring time, temperature, mass, etc. Look up in appropriate references the relationship between units used to measure the same idea (years and days), and the source of the original unit, both historically and in earth phenomena. Group library research followed by reports could be utilized. Discuss how some standard units are conveniently related to other standard units (gram to 1 c.c. of water). Discuss how the inconvenient relation of units is based on the lack of logical considerations (mile and feet) or on historical number systems (base of 60 used in time and angle).	Some standard units are day (based on time of rotation of the earth), gram (mass of 1 cubic centimeter of water), mach (speed of sound), foot candle (intensity of light 1 foot from a defined standard candle), meter (one ten-millionth the distance from the equator to the North Pole).

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-30. Measurement. Tools.	Familiarize students with common instruments.	What common instruments are used in measuring? Obtain instruments to study and practice using. (Ruler, measuring cup, thermometer, light meter, pan balance, spring balance, etc.) In how many cases is length the quantity actually measured directly? (Most.) How can conclusions be drawn about another quantity? The latter is related to a length. Examples: (1) Scale on a measuring cup or baby bottle shows volume since the area of the base does not change. (2) Temperature scale measures length which is related to volume (as above) and the change in volume of which depends on temperature. (3) Light meter measures current, the amount of which released by a metal depends on the light striking it. (4) Ammeter shows the amount of deflection (in angular regions) which depends on the magnetic attraction which in turn is related to the amount of electrical current. (5) The length of arc of a circle of unit radius is directly related to measure of the angular region. Must the relationship of the linear scale and the quantity measured be determined and calculated when using the instrument? (Answer: Usually the calculations have already been done and the scale marked to avoid repeating the calculations.)	Most measurement is indire Most measurement uses tools in which length (which is actually measured) is relat to the quantity to be meast The scales on most measurin instrument are scaled directly in terms of the quantity which is measured indirectly. Area is one of the few quantities for which no tool exists where the measure cannot be read on a scale.

TOPICAL OUTLINE	PURPOSE OR OBJECTIVE	TEACHING APPROACH AND EXAMPLES	SPECIFIC UNDERSTANDING FOR STUDENT
J-31. Statistics. Measures of central tendency.	Analyze data.	Have students measure the length of the room with foot rules. List the measurements by different students. Examples: 24 ft. 24 ft. 24 ft. $1\frac{1}{4}$ in. 23 ft. $10\frac{1}{2}$ in. 24 ft. 1 in. 24 ft. 24 ft. 3 in. 20 ft. 6 in. 24 ft. 1 in. What is the actual length, base on the data calculated? If someone else measures the same length, what measurement would you predict that he would obtain? Discuss various kinds of "averages" and the advantages and disadvantages of each.	Masses of data must be summarized to be studied. An "average" or central tendency is a convenient measure of data and is a good prediction of future scores. A mode is the most frequent score or number. A median is a score or number which is exceeded by exactly one half the scores (i.e. The middle score when the scores are arranged in asc nding or descending order.) A mean is the quotient of the sum of the scores divided by the number of scores. Means are misleading "averages" if the data includes a few extremely high or extremely low scores.

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